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Aspects of Gravity in Quantum Field Theory

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University of Sussex

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Declaration

The work presented in this thesis is based on the results of two publications and one paper which has not yet been submit to a journal. Chapter 2 is based on: Xavier Calmet and Ting-Cheng Yang, Phys. Rev. D **84**, 037701 (2011). Chapter 3 is based on: Xavier Calmet and Ting-Cheng Yang, Int. J. Mod. Phys. A **28**, 1350042 (2013). Chapter 4 is based on unpublished work.

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

Signature:

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UNIVERSITY OF SUSSEX

TING-CHENG YANG, DOCTOR OF PHILOSOPHY

ASPECTS OF GRAVITY IN QUANTUM FIELD THEORY

SUMMARY

This thesis studies three aspects of gravity in quantum field theory. First quantum gravity effects are investigated using effective field theory techniques. In particular, we consider quantum gravity effects in grand unified theory and study their effects on the unification of the masses in such models. We find that the fermion masses unification conditions receive a sizeable correction from the quantum gravitational effects and one thus cannot predict the high energy unification only by the extrapolation from low energy physics without the understanding of gravitational effect in high energy. Secondly we study quantum field theory in curved spacetime in order to understand further about some of the properties of gravity. Keeping gravity as background field we discuss modified gravity theories in different set of parameters called frames; they are the Jordan frame and the Einstein frame respectively. We show how to map gravitational theories at the quantum field theoretical level. The key observation is that there is a non-trivial Jacobian. It can be interpreted as boundary term. Finally we investigate a new canonical quantisation paradigm. In that framework, quantum gravity is power counting renormalisable. Furthermore, the theory is unitary and the problem of time is solved. We use this framework to calculate the solution for the quantum wave function and the semiclassical Hamilton-Jacob function. We study the Hawking-Bekenstein entropy in the spherical symmetric mini-superspace for Schwarzschild black hole, and find that it can be produced naturally from first principles. Importantly it is accompanied naturally by non-thermal quantum correction terms which is generally believed to restore the information loss.

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Notations and Conventions

- Lower case Latin indices represent spatial indices and upper case letters characterise four-dimensional spacetime indices.
- \mathcal{L}_ξ stands for the Lie derivative
- \mathcal{L} stands for Lagrangian
- c is the speed of light
- \dot{q} means time derivative of q . i.e. $\partial_0 q$
- Ψ is the quantum wave function.
- ϕ represents the scalar field.
- α, θ, ϕ are the 3-dimensional spherical coordinates i.e. $\vec{x} = \vec{x}(\alpha, \theta, \phi)$
- Λ is the cosmological constant.
- ξ denotes a Killing vector.
- \sqrt{q} is the square root of determinant of q_{ij} which is a tensor density of weight +1 in three dimensions.
- $\sqrt{-g}$ symbolises the minus of square root of determinant of $g_{\mu\nu}$.
- A quantum operator is characterised with $\hat{\cdot}$. But for convenience the $\hat{\cdot}$ symbol is omitted when there is no ambiguity.
- The signature convention in this thesis is kept $(+, -, -, -)$ for most calculation unless otherwise specified in Chapter 4 for the calculation convenience; this is consistent with most of particle physics calculation.
- We set the speed of light $c = 1$ in most case and specify explicitly in some discussion.

Chapter 1

Introduction

Gravity is the only fundamental force which universally couples to all the fields. Both bosonic and fermionic fields are influenced by gravity in terms of either its presence as a background geometry or its direct coupling to the energy-momentum tensor as the graviton in the semiclassical regime and quantum level. Amongst the four fundamental forces in nature, gravity is the only one for which we do not yet have an established quantum mechanical description. The clash between gravity and the other forces of nature comes from the quantum field theoretical nature of the latter ones. Unfortunately because of the nature of quantum theory, it is well-known that there are difficulties for any of the so far known quantum theories of gravity; perturbative renormalisability [2–9], unitarity loss [10] and the problem of time [11–13] are all well known obstacles for quantum gravity models. The goal of this thesis is to study some aspects of gravity in field theory and then to consider a new approach to quantum gravity which potentially addresses the above issues.

The desire to have a unified theory to describe *all* the interactions including gravity in the same framework is the main motivation to find a quantum theory of gravity [14]. Particle physics which is described by a quantum field theory has already been able to unify three interactions. Such models are known as grand unified theories [15]. Reductionism [16], or the quest for simplicity, has been the main motivation to study the unification of the forces. Typically, models of quantised gravity attempt to overcome at least one of the three obstacles mentioned above. Often, the lack of renormalisability is taken as the most important issue to tackle, but we will also be confronted with the other two problems as we progress through this thesis. We will study a newly proposed paradigm of quantum theory of gravity [1] which could address these issues. Another conceptual reason is that any quantum object with mass will naturally produce a gravitational field

and the superposition of the wave function of this object will need to take into account the superposition of the gravitational field, which of course requires to understand gravity at the quantum level [14]. This is also related to quantum cosmology; if gravity is treated as a quantised field, since it cannot decouple from other fields, the universe must have a wave function. Although quantum cosmology is not our focus we will study the quantum wave function of a generic quantum gravity theory, which may be restricted to the specific model of cosmology. Last but not least, the problem of time lies in the different role of time in quantum mechanics and in general relativity. In quantum mechanics, time is an absolute independent structure and not treated as a dynamical object; yet in general relativity time is treated as a dynamical quantity and there is no absolute reference background. When quantisation applies one would wish to treat time on the same footing as spatial degrees of freedom as in general relativity. However, the fundamental elements of quantum mechanics, such as probability density, are defined in a certain instance of time and all the dynamics, such as the transition amplitude or decay rate in the quantum field theory, are based on the change of physical quantities with respect to an absolute background time. There is no such absolute structure in the context of general relativity. Thus the question is how one can find a good reconciliation of these two perspectives in quantum gravity; this issue is related to the fundamental symmetry in the quantum gravity regime.

As mentioned previously, the aim of this thesis consists in studying quantum aspects of gravity. We start by considering how gravity can impact on Grand Unified Theories (GUTs). GUTs are a framework which enable us to unify three fundamental interactions: electromagnetism, the weak and strong interactions. Usually gravity is not included into GUT models. However, we will show that quantum gravitational threshold effects can be important.

Without having to commit to a specific approach to quantised gravity, one can treat quantum gravitational effects below the Planck scale using effective field theory techniques [17–23] in 2.4. The method depicts the behaviour of physics at high energy level by higher order operators. At energy scales comparably lower than the Planck scale, effective field theory allows operators to be present in terms of an energy expansion obeying the symmetries of the system. This is a sensible way to manifest quantum properties of gravity in the low energy regime. It is a useful tool and can provide us some guidelines for the ultimate quantum theory of gravity. The energy scale we are concerned with is the the GUT scale, 10^{16} GeV. This is close enough to the scale for gravity to consider quantum threshold corrections. Furthermore, because of the Renormalisation Group Equation (RGE), the

Planck mass may be changed [24–26]. It can be even closer to the GUT scale and this necessitates GUTs to take into account the gravity effects. This study is in the context of flat space quantum field theory (QFT) since GUTs are valid in such a context. Moreover, the energy scale concerned is not high enough for gravity to distort the geometry at such a small scale that the quantisation of spacetime is important.

- Our main contribution to this thesis in Chapter 2 is the following: Quantum gravitational threshold corrections to the unification of fermion masses in Grand Unified Theories are calculated explicitly. We show that the running of the Planck mass due to the RGE can have a sizeable effect on these thresholds. They are thus much more important than naively expected. These corrections make any extrapolation from low energy measurements challenging. This work appears in our published paper [27].

Secondly, we can consider gravity as providing a generic curved background for QFT. The matter fields are quantised, while this non-trivial classical background geometry remains unquantised in this context. It is in the context of curved spacetime QFT that subjects such as cosmology in the early universe often reside. In contrast to particle physics in the collider, it is the curved background that provides some different characteristics from ordinary flat space QFT, such as particle production and non-uniqueness of the vacuum. The curved spacetime QFT is still at energies below the scale that the quantum nature of gravity takes place¹. It usually accounts for a semiclassical description of gravity even before quantisation of gravitational fields occurs. Such a semiclassical regime is regarded as the ‘lowest order’ quantum effect of gravity. Moreover this is the bridge for preparing the understanding of a full theory of quantum gravity. Curved space QFT works well at all scales below the scale where the quantum nature of gravity becomes important, for example, away from the proximity of black hole singularities. Thus this study serves as our second step before investigating a full quantum theory of gravity.

In this section we are especially interested in the issue of the dynamical equivalence problem - frame dependence. This is a long standing puzzle in modified gravity theories where gravity couples to scalar fields non-minimally. It is debatable whether the theories in the two sets of formulations (usually called frames), related by a field redefinition of variables are equivalent or not. These two frames are called the Jordan frame and the Einstein frame. This is a problem at both the classical and quantum level. At the classical

¹Scenarios such as extra dimensions, to lower the Planck scale such that quantum gravity effects might appear at the TeV scale in collider experiments, will not be discussed here.

level this has been studied extensively and it is reviewed in Section 3.2. However at the quantum level, this issue has not been studied thoroughly. More studies are required to understand this issue. We therefore look into this issue in curved space QFT.

- Our main contribution to this in Chapter 3 is the following: We found that at the quantum level there is a non-trivial Jacobian coming from the measure of the path integral after the field redefinition. This occurs only at the quantum level. This quantity can be formulated as the expectation value of the stress tensor and it can be trivially normalised in flat space by normal ordering. However in curved spacetime it is highly non-trivial to regulate its divergence. Once this is calculated, we conclude that in order to map the physical result of these two frames, one will have to take into account the Jacobian. The Jacobian can also be interpreted as boundary terms incurred from the frame transformation in our discussion. This result appears in our published paper [28].

Finally in this thesis we discuss the full quantisation of gravity. In order to truly understand the behaviour of gravity at the Planck scale, the proper quantum gravity paradigm is required. There are several established approaches to quantise gravity. According to the categorisation of quantum theories of gravity [29] mentioned in [14], there are two types of quantum gravity theories. The first is the ‘primary theory of quantum gravity’ usually understood as a bottom up approach. From a given classical theory one applies heuristic quantisation rules, such as covariant approaches, path integral, or canonical approaches. The former preserves four dimensional covariance to the quantum level while the latter separates space and time degrees of freedom (at least) in the classical level. Canonical approaches can possibly give a reasoning for the non-trivial meaning of time in the quantum regime as we shall see later. These approaches are traditionally how the successful quantum electrodynamics (QED) works. Applied to gravity, this approach is usually referred to as quantum general relativity or quantum geometrodynamics. The benefit of this method is that the foundation of the theory is known and the quantisation method is already given. There are examples like conventional canonical quantisation of gravity [30, 31], path integral quantisation [2–9], dynamical triangulation [32, 33], and asymptotically safe gravity [34–40]. This approach is less speculative but not guaranteed to offer a unification description. The ‘secondary theory of quantum gravity’, is usually a top-down approach. One begins with a fundamental high energy framework with certain new degrees of freedom, which have already integrated gravity; the goal is to reproduce the lower energy physics. String theory; see, for example, [41–48] is the typical example and embraces extra advantages such as unification. Yet it starts totally from speculation and connection with Standard Model is a challenge. Also loop quantum gravity [49–55] can be categorised as a secondary quantum theory of gravity since it adopts new degrees

of freedom as a canonical pair, even though it mainly deals with canonical quantisation of gravity.

In this study we adopt conventional canonical quantisation, a primary theory of quantum gravity². It is one of the well recognised minimal extension approaches to quantise gravity. Like other quantum gravity theories it confronts some challenges - the problem of time, unitarity and renormalisability. Furthermore there are some problems of the canonical quantisation framework itself which we need to address. The arena for quantum gravity in this approach is called superspace which is a space allowing all possible configurations of three geometry at each point. As in conventional canonical quantisation, one imposes canonical quantisation rules to the Dirac algebra; thus the metric is quantised. Therefore it is a scheme which genuinely quantises the spacetime geometry metric. For this reason it is known as quantum geometrodynamics.

According to the above-mentioned three issues, we study a newly proposed paradigm [1]. It could address potentially these three problems in the same framework. It is a paradigm that inherits power counting renormalisability with unitarity preservation from Hořava's proposal [10] and furthermore solves its problematic ghost mode [56] and inconsistency of algebra. It is distinguished by the separation of one degree of freedom as intrinsic time, and this degree of freedom will be preserved from the quantum regime to the semiclassical regime. In addition the ADM time will be emergent at low energies.

- Our main contribution in Chapter 4 is as follows: In order to make a practical calculation, we restrict the superspace to the spherical symmetric minisuperspace. Overcoming the mathematical difficulties, we found the analytic solution for the quantum wave function of this new quantum gravitational theory.

The rest of this thesis is structured as follows. There are three aspects of study in the quantum field theoretical aspects of gravity. The first appears in Chapter 2: We start with some background about GUTs in Section 2.2 and the running of Planck mass from the renormalisation group equation. Then we calculate the fermion masses in GUTs under the influence of quantum gravity represented by dimension 5 operators taking into account the running of the Planck mass in Section 2.4.

In Chapter 3, it presents the study of the frame dependence issue in curved spacetime QFT. It is structured as follows: We first present the background of the frame dependent

²By 'conventional', we mean that it is the established metric representation without introducing new degrees of freedom as is done in loop quantum gravity. This also falls under the remit of a minimal extension in searching for a new paradigm.

problem in Section 3.2 and some background of curved spacetime QFT in Section 3.4 including transformations of the vacuum state, adiabatic expansion and importantly the renormalisation of the expectation value of the stress tensor. We then detail the classical formalism of the theories we concern in Section 3.3. We preserve the existence of boundary terms because they will provide some intriguing results. Afterwards we focus on our main result at the quantum level in Section 3.5. We explore the Jacobian from the path integral measure and formulate it as the expectation value of the renormalised stress tensor. The importance of the boundary terms is revealed when they are related to the expectation value of the stress tensor in Section 3.6.

Finally, the last topic as Chapter 4 begins with a review of the conventional canonical quantisation of gravity and of Hořava's gravity. The details of the formalism are presented in Section 4.3. We make a symmetry reduction in this new framework to a spherical symmetry in Section 4.4. Using this result, the quantum wave function can be calculated within minisuperspace in Section 4.5.

Chapter 2

Gravitational Corrections to Fermion Masses in Grand Unified Theories

2.1 Introduction

In this chapter we study the first of the three aspects of gravity discussed in the introduction: particle physics with gravitational threshold corrections. Grand unified theories (GUTs) are the arena of this study as they are the paradigm which already accommodates three of the fundamental forces. It is sensible to discuss the influence of gravitational effects on GUTs as it may shed light on the goal to unify all the interactions. Although at first glance the GUT energy scale is not close to that of the quantum gravity scale, as we will see later the Planck scale can be lowered and may even approach the GUT scale by renormalisation group equations (RGEs).

It has been argued that the LHC data could be used to reconstruct, using renormalisation group techniques, the fundamental Grand Unified Theory, see e.g. [57], or differentiate between different supersymmetry breaking patterns [57]. In [58, 59] it has been shown that there are potentially sizeable quantum gravitational corrections to the unification conditions for the gauge couplings of the Standard Model. The thresholds have been known for a while [60–63], but it had not been realised that they could potentially be larger than the two-loop corrections [58]. The aim of this work is to show that this quantum gravity blur has a similar effect on the unification conditions for the masses of the fermions in a grand unified framework. It is appropriate to discuss these results in the context of flat space QFT, like ordinary particle physics, since GUTs are definitely valid

in flat space and the geometry is not distorted due to gravity at the scale of concern.

Here we study quantum gravitational threshold corrections to the unification of fermion masses in Grand Unified Theories. We show that the running of the Planck mass can have a sizeable effect on these thresholds which are thus much more important than naively expected. These corrections make any extrapolation from low energy measurements challenging.

This chapter is structured as follows: After some introduction of grand unification in Section 2.2, the running of Planck mass is derived by the conventional heat kernel method in Section 2.3. This provides an important factor η which depends on the number of fields in the model. It is this factor that has an influence on calculations of quantum threshold corrections from gravity to the gauge unification and fermion masses. The main results are presented in Section 2.4.

2.2 Grand Unification

There are many good reviews on the motivation for GUTs, see for example [64], and so we will not discuss this at length. Gauge invariance provides the guiding principle for particle physics in the standard model. It is clear that quantum chromodynamics, weak and electromagnetic interactions are all accounted for by gauge theories. It is a natural extension beyond the stand model that a simple gauge group provides unification such that the interactions will be determined by less constants - in the spirit of theoretical physics, one wishes to describe nature with as few free parameters as possible. The scale associated with the larger symmetry group for unification, is set by the where running of coupling constants approximately coincides and is typically assumed to be at around 10^{16} GeV. The quantum fields of the Standard Model fit nicely into simple representations of a Grand Unified Theory [15] such as e.g. $SU(5)$ or $SO(10)$. The idea of unification is extremely attractive for several reasons. As mentioned above, grand unification drastically reduces the number of independent coupling constants. Furthermore, when extrapolated using renormalisation group equations, the value of the strong and electroweak interactions measured at low energy seem to converge amazingly to some common value at around 10^{16} GeV [65–67] if the Standard Model is replaced by the Minimal Supersymmetric Standard Model at around a TeV. An important feature of Grand Unified Theories is that they predict the existence of many, potentially heavy, new particles. This is due to the very nature of Grand Unified Theories which need to be based on groups large enough to incorporate the Standard Model $SU(3) \otimes SU(2) \otimes U(1)$ such that $G \supset SU(3)_c \otimes SU(2)_L \otimes$

$U(1)_Y$. In addition, grand unified theories often incorporate multiplets with a large number of fields to obtain viable phenomenology. When the unified theory is supersymmetric, the number of fundamental fields is even larger. The large number of fields will play an important role in our following studies as shown later.

While our study can be adopted into other GUT models and the choice of unification group G depends on the question being addressed, without losing generality we take $SU(5)$ grand unification as a prototype for this study because it is the minimal unification extension from the standard model. In the $SU(5)$ GUT, we only invoke the information required for the calculation. The first generation of fermions can be assigned to the $\mathbf{5}^*$ and $\mathbf{10}$ respectively:

$$\mathbf{5}^* = f_L = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ -\nu_e \end{pmatrix}_L \quad (2.1)$$

$$\mathbf{10} = \Psi_L^{jk} = \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{pmatrix}_L. \quad (2.2)$$

These will be used in the calculation later on.

2.3 Running of Planck Mass

The unification scale of concern is at around 10^{16} GeV, which appears to be far away from the quantum effects of gravity. However, an important consequence of the large number of fundamental fields in the unification theories, which can easily reach 1000, is that the scale at which quantum gravitational effects are expected to become large is not necessarily as expected at some 10^{19} GeV but is given by the renormalised Planck mass probed at the energy scale μ [24–26]:

$$M(\mu)^2 = M(0)^2 - \frac{\mu^2}{12\pi}(N_0 + N_{1/2} - 4N_1) \quad (2.3)$$

where $M(0)$ is the Planck mass at low energy, i.e. Newton's constant is given by $G = M(0)^{-2}$ in natural units, and N_0 , $N_{1/2}$ and N_1 are respectively the numbers of real scalar fields, Weyl spinors and spin one vector bosons.

If the strength of gravitational interactions is scale dependent, the scale μ_* at which quantum gravity effects are large is the one at which

$$M(\mu_*) \sim \mu_* . \quad (2.4)$$

It has been shown in [58, 59] that the presence of a large number of fields can dramatically impact the value μ_* . In many Grand Unified models, the large number of fields can cause the true scale μ_* of quantum gravity to be significantly lower than the naive value $M_{\text{Pl}} \sim 10^{19}$ GeV. In fact, from the above equations, one finds

$$\mu_* = \frac{M_{\text{Pl}}}{\eta} , \quad (2.5)$$

where, for a theory with $N \equiv N_0 + N_{1/2} - 4N_1$,

$$\eta = \sqrt{1 + \frac{N}{12\pi}} . \quad (2.6)$$

In order to understand the details of the running of the Planck mass, we need to invoke a specific technique - namely the heat kernel. This technique is analogous to adiabatic expansion [68] and the DeWitt-Schwinger representation for Green's functions [69–71]. We will discuss this result in the next section before moving on to the main result of this chapter.

2.3.1 Derivation of the Running of Planck Mass

For completeness we use a pedagogic method for a brief derivation of the above consequences based on the formula in [68, 72]. The generating function is defined as¹

$$Z[J] = \int \mathcal{D}\phi \exp \frac{i}{\hbar} (S + \int dV J \phi) \quad (2.7)$$

$$Z[0] = \langle \text{out}, 0 \mid \text{in}, 0 \rangle \quad (2.8)$$

$$= \int \mathcal{D}\phi \exp \frac{i}{\hbar} \int \frac{1}{2} dV (\delta_{ij} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j - m_{ij}^2 \phi^i \phi^j) \quad (2.9)$$

$$= \int \mathcal{D}\phi \exp \frac{i}{\hbar} \int \frac{1}{2} dV (-\phi^i D_{ij} \phi^j) \quad (2.10)$$

$$\equiv \exp(\frac{i}{\hbar} W) \quad (2.11)$$

² with $dV = \sqrt{-g} d^4x$. $D_{ij}(x, x') = [\delta_{ij} \square_x + m_{ij}^2(x)] \delta(x, x')$ the differential operator and $m_{ij}(x) = m^2 \delta_{ij}$ for minimally coupled scalar fields³ which leads to $\int dV D_{ij}(x, x') G^{jk}(x', x'') =$

¹Here we include scalar field only for simplicity; it can be extended to other fields.

² δ_{ij} is the flat metric for the internal ‘field space’ which is the space spanned by the scalar field ϕ^i as defined in Section 3.2. This is also the standard convention and so is normally not specified explicitly.

³ $m_{ij}(x) = (m^2 + \xi R) \delta_{ij}$ for non-minimally coupled scalar fields.

$\delta_i^k \delta(x, x'')$ with $G_{ij}(x, x')$ as the Green's function. In terms of DeWitt condensed notation [30, 69] with indices including both the spacetime arguments and field labels and summation of indices including the associated spacetime integration this reads $D_{ij'} G^{j'k''} = \delta_i^{k''}$. This also implies $G^{ij'} = -(D^{-1})^{ij'}$. Through the Schwinger variational principle [73] one can obtain

$$\delta Z[0] = \delta \langle \text{out}, 0 \mid \text{in}, 0 \rangle \quad (2.12)$$

$$= \frac{i}{\hbar} \int \mathcal{D}\phi \delta S \exp \frac{i}{\hbar} S \quad (2.13)$$

$$= \frac{i}{\hbar} \langle \text{out}, 0 \mid \delta S \mid \text{in}, 0 \rangle \quad (2.14)$$

and therefore

$$\delta W = \frac{\langle \text{out}, 0 \mid \delta S \mid \text{in}, 0 \rangle}{\langle \text{out}, 0 \mid \text{in}, 0 \rangle}. \quad (2.15)$$

Because $\frac{\langle \text{out}, 0 \mid \phi^i(x) \phi^i(x') \mid \text{in}, 0 \rangle}{\langle \text{out}, 0 \mid \text{in}, 0 \rangle} = -i\hbar G(x, x')$ is the two point Green's function, and because the metric is not varied $\delta m_{ij}^2(x, x')$ is actually $\delta D_{ij'}$, we find

$$\delta W = \frac{i\hbar}{2} \int dV dV' [\delta m_{ij}^2(x, x')] G^{ji}(x', x) \quad (2.16)$$

$$= \frac{i\hbar}{2} \text{Tr}[\delta m_{ij'}^2] G^{j'k''} \quad (2.17)$$

$$= \frac{i\hbar}{2} \text{Tr}[\delta D_{ij'}] (D^{-1})^{j'k''} \quad (2.18)$$

$$= \delta \left[\frac{i\hbar}{2} \text{Tr} \ln(l^2 D_{ij'}) \right] \quad (2.19)$$

where l^2 has dimensions of length squared for the purposes of correct dimensionality and the mathematical equality for any operator $A^{-1} \delta A = \delta \ln(l^2 A)$. Then the generating function W can be expressed in the following form

$$W = -i\hbar \ln Z[0] \quad (2.20)$$

$$= \frac{i\hbar}{2} \text{Tr}[\ln(l^2 D_{ij})] \quad (2.21)$$

$$= \frac{i\hbar}{2} \ln[\det(l^2 D_{ij})] \quad (2.22)$$

$$= -\frac{i\hbar}{2} \text{Tr}[\ln(-G_{ij})]. \quad (2.23)$$

By virtue of the Feynman propagator, the inverse of the differential operator is the Green's function, which can be written with a small negative imaginary part in order to give the correct integration contour as,

$$D^{-1} = i \int_0^\infty d\tau \exp(-i\tau D), \quad (2.24)$$

we omit the indices for the moment. Then

$$\delta W = \frac{i\hbar}{2} \text{Tr}[\delta D](D^{-1}) = \frac{i\hbar}{2} \text{Tr}\left[\int_0^\infty d\tau i \exp(-i\tau D)\right] \delta D = -\frac{i\hbar}{2} \delta \int_0^\infty \frac{d\tau}{\tau} \text{Tr} \exp(-i\tau D),$$

which implies the form of W is

$$W = -\frac{i\hbar}{2} \int_0^\infty \frac{d\tau}{\tau} \text{Tr} \exp(-i\tau D). \quad (2.25)$$

Then we define the heat kernel function

$$K_j^i(\tau; x, x') = \sum_n e^{-i\tau\lambda} f_n^i(x) f_{nj}^*(x'), \quad (2.26)$$

with f_{nj} as the normalised eigenfunction and λ_n as the eigenvalue of the differential operator $D_{ij}(x, x') f_n^j(x') = \lambda_n f_{ni}(x)$ and $\int dV f_n^i(x) f_{mi}^*(x) = \delta_{nm}$. Therefore $\text{Tr}(\exp(-i\tau D)) = \int dV K_i^i(\tau; x, x)$ can be understood straightforwardly by the definition of the heat kernel function. Using this in our generating function W yields

$$W = -\frac{i\hbar}{2} \int dV \int_0^\infty \frac{d\tau}{\tau} \text{Tr} K(\tau; x, x). \quad (2.27)$$

The reason K_j^i is coined as the heat kernel function is because it obeys the Schrödinger equation when applying the eigenvalue value equation $D_{ij}(x, x') f_n^j(x') = \lambda_n f_{ni}(x)$ to (2.26):

$$i \frac{\partial}{\partial \tau} K_j^i(\tau; x, x') = D_l^i K_j^l(\tau; x, x'); \quad (2.28)$$

with the boundary condition $K_j^i(0; x, x') = \delta_j^i \delta(x, x')$, which can also be obtained from its definition (2.26). This equation (2.28) is the diffusion or heat equation where τ is analytically continued to an imaginary value. Please notice the heat kernel function is also proportional to the Green's function by virtue of (2.23).

The advantage of the heat kernel function resides in the the fact that it can have an asymptotic expansion for small τ , which corresponds to the order of the adiabatic expansion in [68], this expansion is originally derived by [70, 74] following the work of [71],

$$K(\tau; x, x) \sim i(4\pi i\tau)^{-\frac{n}{2}} \sum_{k=0}^{\infty} (i\tau)^k E_k(x) \quad (2.29)$$

where n is the number of spacetime dimensions. In the context of regularisation, the large τ of the asymptotic expansion provides the heat kernel function with exponential decay and the divergence only comes from the region between $\tau = 0$ and the small value of τ_0 ; accordingly we can separate out the divergent part as

$$\begin{aligned} \text{div} W &= -\frac{i\hbar}{2} \int dV \int_0^{\tau_0} \frac{d\tau}{\tau} \text{Tr} K(\tau; x, x) \\ &= \frac{i\hbar}{2} (4\pi)^{-\frac{n}{2}} \int dV \sum_{k=0}^{\infty} \text{Tr} E_k(x) \int_0^{\tau_0} d\tau (i\tau)^{k-1-\frac{n}{2}}. \end{aligned} \quad (2.30)$$

Exploiting this property of the heat kernel function one can regularise the one loop generating function and therefore renormalise the coupling constant - Newton's constant in our case. The consequent heat kernel function is [24, 26, 75]:

$$K(\tau, x, x) = \frac{1}{(4\pi\tau)^2} \left\{ \int d^4x \sqrt{-g} + \frac{\tau}{6} \int d^4x \sqrt{-g} R + \mathcal{O}(\tau^{\frac{3}{2}}) \right\}. \quad (2.31)$$

Substituting this into (2.27) and comparing with the Einstein Hilbert action with minimally coupled scalar fields and without a cosmological constant,

$$S = S_g + S_\phi = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R + \frac{1}{2} (\delta_{ij} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j) \right). \quad (2.32)$$

One can obtain the renormalised Newtonian constant by the contribution from scalar fields and similarly from the spinorial and vector fields [24–26]. Combine these result together, it is

$$\frac{1}{G(\mu)} = \frac{1}{G(0)} - \frac{\mu^2}{12\pi} (N_0 + N_{1/2} - 4N_1) \quad (2.33)$$

where N_0 , $N_{1/2}$ and N_1 are respectively the number of real scalar fields, Weyl spinors and spin one vector bosons producing the running of Planck mass (2.3). Notice that we analytic continue to the Euclidean space and define the heat kernel function as $W = -\frac{1}{2} \int^{\mu^{-2}} d\tau \text{Tr} \frac{K(\tau, x, x)}{\tau}$ with the infrared cutoff μ .

2.4 Gravitational Corrections to Fermion Masses in Grand Unified Theories

Effective field theories allow us to study the physics at low energies with the inclusion of the effects of physics at higher energy, which may comprise some unknown new degrees of freedom, see [17–23] for example. Taking gravity for example, it is not perturbatively renormalisable beyond a certain cut-off scale and it is regarded as a fact that there are new degrees of freedom which correctly describe gravity beyond this regime, as in string theory for example. Precisely, the problem is that divergences beyond two loops for pure gravity [2–9] cannot be absorbed by a finite number of coupling constants; therefore the theory loses predictability. This means that the two loop divergence [76, 77] containing the gravitational constant G can not be re-absorbed by a local field redefinition. However, even before we have consensus on how to find the correct quantum theory of gravity, the high energy behaviour can manifest itself at low energy scales by effective field theory (EFT). EFT can separate the high energy degrees of freedom, and thus ultraviolet divergences, by integrating out beyond a certain energy level. This leaves local operators expanded in terms of an energy scale - curvature or number of derivatives - with the respective coupling constants or coefficients needing to be determined and constrained by experiment. The operators in the energy expansion need to obey the symmetries in the context being considered, such as general covariance and gauge invariance. The operators form a perturbative expansion with higher order terms being suppressed by higher powers of the high energy scale; thus one can calculate up to a desired order of accuracy. For example, in the case of gravity, the next to leading order operator is the one with two powers of the scalar curvature and so on. Thus $S_{eff} = \int d^4x \sqrt{-g} \{ \Lambda + \frac{c^3}{16\pi G} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \dots \}$ has its derivatives ordered as ∂^0 , ∂^2 and ∂^4 in each term. The effects of the high energy behaviour beyond the cutoff scale will be embedded into these coupling constants. EFT is a sensible method to study the quantum behaviour of gravity in the low energy regime. By virtue of EFT a non-renormalisable theory can be used to calculate to the loop order desired by utilising the required terms in the operator expansion and the corresponding coupling constants.

This technique has been well proven in contexts where the correct full theory exists and the results from effective field theory are fully compatible with the full theory to the order of accuracy calculated, see for example [78–81]. This property will still hold for theories without an ultraviolet completion, which means a theory is only confirmed for some low energy behaviour, as is the case for gravity. This is because the EFT allows

us to study a theory below a certain energy scale with the relevant degrees of freedom and symmetries only and the high energy behaviour emerges as the effective higher order terms in the expansion, as mentioned above. Thus the low energy behaviour is reliable. The standard procedure to construct an EFT is to write down the most general operators in the Lagrangian which contain the allowed symmetries and particles. The operators are then usually ordered in an energy expansion. In what follows we will only consider the lowest order operators. In the EFT one allows canonical dimension greater than four. In our study the lowest order is dimension five which will provide the leading effect.

In [58], quantum gravity effects have been shown to affect the unification of gauge couplings (see [60–63, 82–87], for a non-exhaustive list of papers). The *lowest order* effective operators induced by a quantum theory of gravity are of dimension five, such as [60, 61]

$$\frac{c}{\hat{\mu}_*} \text{Tr} (G_{\mu\nu} G^{\mu\nu} H), \quad (2.34)$$

where $G_{\mu\nu}$ is the GUT field strength and H is a scalar multiplet and

$$\hat{\mu}_* = \mu_*/\sqrt{8\pi} = \hat{M}_{\text{Pl}}/\eta. \quad (2.35)$$

This operator is expected to be induced by strong non-perturbative effects at the scale of quantum gravity, and so it has a coefficient $c \sim \mathcal{O}(1)$ and is suppressed by the true reduced Planck scale with $\hat{M}_{\text{Pl}} = 2.43 \times 10^{18} \text{ GeV}$.

The importance of gravitational effects were illustrated in [58] using the example of SUSY-SU(5). Operators similar to (2.34) are present in all GUT models and an equivalent analysis applies. The η factor, whose value is model dependent, originates from the particle content in the formula of the running of the Planck mass (2.3) and is a factor in all effective operators and will substantially modify the scale of the Planck mass, especially for more complicated models accommodating large numbers of fields. Usually it lowers the Planck mass and necessitates the quantum corrections from gravity to be taken into account for GUTs.

In SU(5) the multiplet H in the adjoint representation acquires, upon symmetry breaking at the unification scale M_X , a vacuum expectation value

$$\langle \phi(24) \rangle = M_X \text{diag} (2, 2, 2, -3, -3) / \sqrt{50\pi\alpha_G} = \text{diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2})(2\sqrt{2}/5g_u)M_x,$$

where α_G is the value of the SU(5) gauge coupling at M_X and where $g_u = \sqrt{4\pi\alpha_G}$. Inserted into the operator (2.34), this modifies the gauge kinetic terms of $\text{SU}(3) \otimes \text{SU}(2) \otimes \text{U}(1)$ below

the scale M_X to

$$-\frac{1}{4}(1+\epsilon_1)F_{\mu\nu}F_{\text{U}(1)}^{\mu\nu}-\frac{1}{2}(1+\epsilon_2)\text{Tr}\left(F_{\mu\nu}F_{\text{SU}(2)}^{\mu\nu}\right) \\ -\frac{1}{2}(1+\epsilon_3)\text{Tr}\left(F_{\mu\nu}F_{\text{SU}(3)}^{\mu\nu}\right) \quad (2.36)$$

with

$$\epsilon_1 = \frac{\epsilon_2}{3} = -\frac{\epsilon_3}{2} = \frac{\sqrt{2}}{5\sqrt{\pi}} \frac{c\eta}{\sqrt{\alpha_G}} \frac{M_X}{\hat{M}_{\text{Pl}}} . \quad (2.37)$$

After a finite field redefinition $A_\mu^i \rightarrow (1+\epsilon_i)^{1/2} A_\mu^i$ the kinetic terms have a familiar form, and it is then the corresponding redefined coupling constants $g_i \rightarrow (1+\epsilon_i)^{-1/2} g_i$ that are observed at low energies and that obey the usual renormalisation group equations below M_X , whereas it is the *original* coupling constants that need to meet at M_X in order for unification to happen. In terms of the observable rescaled couplings, the unification condition therefore reads:

$$\alpha_G = (1+\epsilon_1)\alpha_1(M_X) = (1+\epsilon_2)\alpha_2(M_X) \\ = (1+\epsilon_3)\alpha_3(M_X) . \quad (2.38)$$

We now study the usual one loop beta function for the running of the gauge and Yukawa couplings to determine the magnitude of the threshold corrections needed to obtain a good numerical unification of the parameters which ought to merge into one value at the unification scale. The initial value mainly comes from [88]. The one loop renormalisation group equations of the gauge couplings of the standard model are given by

$$\frac{1}{\alpha_i(t)} = \frac{1}{\alpha_{0i}} + \frac{b_i}{2\pi}(t-t_0) \quad (2.39)$$

with

$$\alpha_i(t) = \frac{g_i^2(t)}{4\pi} \quad (2.40)$$

where $t = \ln u$ with u the energy scale of interest, with initial value $u_0 = 91.187$ GeV and $t_0 = \ln u_0 = 4.51291$. $b_1 = -41/10$, $b_2 = 19/6$ and $b_3 = 7$. The aim of this exercise is to determine at what energy scale one obtains the unification of the gauge couplings for order one Wilson coefficients. Our results are presented in Table 2.1. We find that for $M_X \sim 10^{17}$ GeV the Wilson coefficient c is of order unity.

Unification scale	$\alpha_1^{-1} = \alpha_2^{-1} = \alpha_3^{-1}$	ε_1	ε_2	ε_3	Wilson Coefficient c
$M_X = 10^{16}$ GeV	34.2513	0.112999	0.338996	0.298127	38.2604
$M_X = 3 \times 10^{16}$ GeV	32.8992	0.136953	0.41086	0.388683	12.7535
$M_X = 10^{17}$ GeV	31.4173	0.165573	0.496719	0.496877	3.82604

Table 2.1: Wilson coefficient for different unification scales M_X .

It was shown in [58] that the effects can be larger than the two loop effects considered in e.g. [67] and that they could either invalidate claims of a perfect unification of a SUSY-Standard Model or on the contrary help to unify models whose gauge couplings would apparently not unify.

In this work we point out that the same physical effect can have important implications for fermion masses. Again we will be using a simple SU(5) model to make our point more explicit, but our results can be trivially generalised to any Grand Unified Theory. One of the most interesting predictions of a Grand Unified Theory, besides the unification of the gauge couplings at the unification scale, is the unification of some of the fermion masses at the unification scale. Fermion masses are generated by the Yukawa interactions. For example, in the simple SU(5) grand unification model with a Higgs in the **5** representation, one has

$$\mathcal{L} = \{G_d \bar{\Psi}_{jR}^c \Psi_{kL}^j H^k(5) + G_u \varepsilon_{jklmn} \bar{\Psi}_L^{cjk} \Psi_L^{lm} H^n(5)\} + h.c. \quad (2.41)$$

$$= -\frac{2M_w}{\sqrt{2}g_2} [G_d(\bar{d}d + \bar{e}e) + G_u 8(\bar{u}u)] \quad (2.42)$$

and one obtains

$$m_d(M_X) = m_e(M_X) = -\frac{2M_w}{\sqrt{2}g_2} G_d \quad (2.43)$$

where M_w is the W -boson mass, g_2 the SU(2) gauge coupling and G_i are Yukawa couplings. This is one of the most exciting results of Grand Unified Theories, namely at the unification scale M_X the masses of the down-type quarks are equal to the masses of the charged leptons, while the mass of the u -type quarks are not related to other parameters of the model. The up-type quark masses are given by $m_u(X) = -\frac{16M_w}{\sqrt{2}g_2} G_u$ at the unification scale.

In analogue to (2.34), there are also dimension five operators which can affect the fermion masses. They have been considered a while ago by Ellis and Gaillard [89] (see also [90])

$$\frac{c}{\hat{\mu}_\star} \bar{\Psi} \phi \Psi H + h.c. \quad (2.44)$$

where Ψ are fermion fields, ϕ and H some scalar boson multiplets chosen in appropriate representations. In a simple SU(5) toy model with scalar fields in the **24** and **5** representations with their vacuum expectation value $\langle \phi(24) \rangle = \text{diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2})(2\sqrt{2}/5g_u)M_x$

and $\langle H(5) \rangle = \text{diag}(0, 0, 0, 0, 1)(2/\sqrt{2}g_2)M_w$ respectively. One gets

$$\begin{aligned}
O_5 = & \frac{a_1}{\hat{\mu}_\star} \{ \phi_{mn} \bar{f}^{mk} H_k^l \Psi_l^n \} \\
& + \frac{a_2}{\hat{\mu}_\star} \{ \phi_{mn} H^{mk} \bar{f}_k^l \Psi_l^n \} \\
& + \frac{a_3}{\hat{\mu}_\star} \{ \phi_{mn} \bar{f}^{mk} H_k^l \Psi_l^n \} \\
& - \frac{a_4}{\hat{\mu}_\star} \{ \phi_{mn} H^{mk} \bar{f}_k^l \Psi_l^n \} \\
& + \frac{a_5}{\hat{\mu}_\star} \varepsilon^{mnpql} \{ \Psi_{mn} \Psi_{pq} H_k \phi_l^k \} \\
& + \frac{a_6}{\hat{\mu}_\star} \varepsilon^{mnpkl} \{ \Psi_{mn} \Psi_{pq} H_k \phi_l^q \}
\end{aligned} \tag{2.45}$$

where Ψ and f are fermion fields in **10** and **5** respectively and $\hat{\mu}_\star$ is from the definition of (2.35). The a_1 and a_2 terms are symmetric while a_3 and a_4 are anti-symmetric. These operators have been studied extensively see e.g. [91] and references therein for more recent works in that direction. However the renormalisation group improvement considered here has not been previously studied. In $SU(5)$, the value of the expectation values of $\phi(24)$ and $H(5)$ are fixed by the requirement that the Grand Unified Theory be broken at some 10^{16} GeV, i.e. $\langle \phi(24) \rangle \sim 10^{16}$ GeV and that the spontaneous symmetry breaking of the electroweak interactions takes place at the weak scale, i.e. $\langle H(5) \rangle = 246$ GeV.

These operators lead to a modification of the unification condition for the down-type quarks and their respective charged leptons. One finds the threshold correction is

$$m_d(M_X)[1 + 2(\zeta_1 + \zeta_2 + \zeta_3 - \zeta_4)] = m_e(M_X)[1 + \frac{9}{2}(\zeta_1 - \zeta_2 - \zeta_3 + \zeta_4)] \tag{2.46}$$

with

$$\zeta_i = \frac{-2\sqrt{2}}{5G_d g_u} \frac{M_X}{M_{Pl}} a_i \eta \tag{2.47}$$

where g_u is the unified coupling constant. The a_6 term vanishes because the anti-symmetric part is multiplied by the symmetric part. We note that u -type quark masses do receive a correction due to one of these operators; the non-vanishing correction is,

$$m_u(M_X)(1 + \frac{3}{8}\zeta_3). \tag{2.48}$$

We now consider the unification of fermion masses. For the Yukawa couplings we use the standard model one loop renormalisation group, approximating the CKM matrix by the identity matrix. The renormalisation group equations for the Yukawa couplings of the different fermions are given by [92],

$$\frac{d\mathbf{Y}_{u,d,e}}{dt} = \mathbf{Y}_{u,d,e} \left(\frac{1}{16\pi^2} \beta_{u,d,e}^{(1)} \right) \tag{2.49}$$

where

$$\beta_u^{(1)} = \frac{3}{2}(\mathbf{Y}_u^\dagger \mathbf{Y}_u - \mathbf{Y}_d^\dagger \mathbf{Y}_d) + Y_2(s) - \left(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right) \quad (2.50)$$

$$\beta_d^{(1)} = \frac{3}{2}(\mathbf{Y}_d^\dagger \mathbf{Y}_d - \mathbf{Y}_u^\dagger \mathbf{Y}_u) + Y_2(s) - \left(\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right) \quad (2.51)$$

$$\beta_e^{(1)} = \frac{3}{2}\mathbf{Y}_e^\dagger \mathbf{Y}_e + Y_2(s) - \frac{9}{4}(g_1^2 + g_2^2), \quad (2.52)$$

where

$$Y_2(s) = \text{Tr}\{3\mathbf{Y}_u^\dagger \mathbf{Y}_u + 3\mathbf{Y}_d^\dagger \mathbf{Y}_d + \mathbf{Y}_e^\dagger \mathbf{Y}_e\} \quad (2.53)$$

and

$$\mathbf{Y}_e = \frac{\sqrt{2}}{\nu} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \quad (2.54)$$

$$\mathbf{Y}_d = \frac{\sqrt{2}}{\nu} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \quad (2.55)$$

$$\mathbf{Y}_u = \frac{\sqrt{2}}{\nu} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} V_{\text{CKM}}. \quad (2.56)$$

The resultant Wilson coefficients calculated from the modified unification condition of the fermion masses discussed above is shown below. In order to calculate, we merge all the Wilson coefficients to be of the same order because the number of Wilson coefficients is more than the number of equations.

Unification scale	a_i for 1st generation	a_i for 2nd generation	a_i for 3rd generation
$M_X = 10^{16}$ Gev	0.0458788	0.0151651	0.0358902
$M_X = 3 \times 10^{16}$ Gev	0.0152929	0.00505504	0.0119634
$M_X = 10^{17}$ Gev	0.00458788	0.00151651	0.00358902

Table 2.2: Wilson coefficients for different unification scales M_X .

Clearly, since the scale $\hat{\mu}_*$, i.e., the effective reduced Planck mass, is very poorly known and depends on the number of fields in the unified theory, it is very difficult to argue that these quantum gravitational effects can be neglected. Based on (2.6), while in this simple $SU(5)$ model, η is only equal to 0.74 as shown in [59] η can easily be as large as 8.1 in SUSY- $SO(10)$ models whose $N \equiv N_0 + N_{1/2} - 4N_1 = 2445$. Though in the $SU(5)$ model the η factor is almost of order 1 and provides no significant change to the Planck mass,

yet this toy model elucidates that the quantum corrections from gravity already provide sizeable modifications to the unification condition. The effect will be more obvious in other models with larger numbers of fields such as SUSY- $SO(10)$ as mentioned. The running of the Planck mass thus has a potentially large impact on the splitting at the unification scale of the down type quarks and down type leptons. It is easy to evaluate the magnitude of the effect. One finds $\zeta_i \sim 10^{-2} a_i / G_e \eta$, where we used $\alpha_u \sim 1/40$ and $M_X / \bar{M}_{Pl} \sim 10^{-2}$. Even if the a_i are as tiny as the corresponding Yukawa couplings, one can get a 10% effect for Grand Unified Theories with a large matter content and thus large η . Once again we see that renormalisation group effects of the Planck mass can have sizeable effects on the unification conditions of Grand Unified Theories.

2.5 Conclusion

There are several implications of these results. Without a precise knowledge of the quantum gravitational corrections, i.e. of the full theory of quantum gravity, it is very difficult to extrapolate from low energy measurements to check whether fermion masses unify or not. This casts some doubts concerning the feasibility of reconstructing the parameters of a Grand Unified Theory by using low energy measurements performed at the Large Hadron Collider. On the other hand, these threshold effects can help to explain the low energy pattern of fermion masses and can revive models which naively would predict the incorrect pattern in the low energy regime. This motivates further understanding for the ultimate quantum theory of gravity.

As a summary, we have reconsidered quantum gravitational threshold effects studied a long time ago by Ellis and Gaillard. We have shown that the running of the Planck mass can have a sizeable effect and that these threshold corrections are much more important than naively expected. This result is in line with our previous observations concerning the quantum gravitational threshold corrections to the unification of the coupling constants of the Standard Model.

Chapter 3

Frame Transformations of Gravitational Theories

3.1 Introduction

This chapter aims to understand the equivalence between the Jordan frame and Einstein frame for modified gravitational theories in the context of curved space quantum field theory. As described before in chapter 1, this represents our second aspect of study in gravitation with generic curved classical background and quantised matter fields. In terms of a more generalised approach to field redefinitions (reparameterisation), we propose a systematic method which can be applied to a wide class of theories related by field reparameterisations. In order to investigate these two frames at the quantum level, the generating functional (partition function) Z , and the Schwinger-Dyson equation will play central roles to the study of the relevant physical quantities. We find that at the quantum level, the Jacobian produced from the measure by the field redefinition must be taken into account. This Jacobian can be represented as the expectation value of the stress tensor and re-expressed as a non-conserved current within a total derivative, thus in a sense it can be considered a boundary term. This expectation value has been well-studied in curved space QFT. Meanwhile the conserved current, and also the quantum field equations of motion, are represented as total derivatives, which can be interpreted as a boundary term. The conserved current and quantum equations of motion are modified by the Jacobian found above. We also propose a method to offset this effect by adding additional boundary terms. Generically, physical observables shall always be unique irrespective of frames, thus if one intends to compare the calculations from one set of variables to another set of variables the extra term from the Jacobian (in a sense a non-trivial boundary term), must be taken

into account on top of the conserved current.

3.2 Review of Dynamical Equivalence

The Einstein Hilbert action can be extended to couple with scalar fields, which are part of the wide class of scalar-tensor theories in modified gravity [93]. These theories are usually related by certain types of transformations. One usually constructs the theory in one set of variables and makes a field redefinition to the other set of variables. Each individual set of variables is conventionally called a ‘frame’. The main purpose of changing frames is to reduce the complexity of calculating the desired physical quantities. Amongst the frames, the Jordan frame is the one in which the gravitational field couples with scalar fields in a non-minimal way while the Einstein frame is the one where such non-minimal couplings are absent. In other words, starting from the Jordan frame, it is always possible to perform a redefinition of the scalar field and the metric tensor to obtain a Lagrangian without the non-minimal coupling. This frame is called the Einstein frame since the coefficient in front of the Ricci scalar is $\frac{1}{16\pi G}$ and thus has the form of the usual Einstein-Hilbert action. While it is often convenient to build a model in the Jordan frame, calculations may appear to be more difficult to perform using these degrees of freedom and frequently relativists and cosmologists transform their models to the Einstein frame to compare with existing inflationary calculations to bound the parameters of their models. The transformation between them is governed by specific field reparameterisations relating the scalar field, metric and all the other derived variables and one can work out the transformation with mathematical consistency; see for example [94–101]. Regardless of whether they are exactly conformal transformations¹ or not, they all count as field reparameterisations. Nonetheless their equivalence from the physical view point has been an open question for a long time; see, for example [102–107]. Authors would either claim equivalence or inequivalence but with a certain frame physically favoured. In [108] there is special interest in the effect of boundary terms. As the issue about dynamical equivalence/inequivalence at the classical level has been discussed for a long time; the issue at the quantum level is also an essential issue and for any modified gravitational theory in different frames in curved space it is still an open question. As we will clarify later, the frame dependence problem occurs only when making a comparison between them. Namely, if one stays in one frame without making a transformation to the other frame and meanwhile also rewrites the variables back to the

¹It means that the transformation regarding the metric and field with the relation both $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and $\tilde{\phi} = \Omega^{-1}\phi$.

original previous variables to make a comparison then there exists no such problem. If one only makes a transformation to a new frame and remains in that frame then there should be no confusion. Accordingly we address this issue by performing the transformation to the new frame and rewriting the variables back to those of the original frame in order to compare and to understand the equivalence. Our focus in this chapter is quantisation of matter fields on a curved background which is regarded as a classical field. Thus it is in the sense of a semiclassical method where the Einstein field equation has the classical field on the right hand side and the matter fields as the expectation value of the stress tensor ².

The appropriate framework for this discussion is curved space time field theory. The reason that we discuss this topic in the context of curved space is as follows: For any field theory in the presence of a gravitational field, if there exists non zero curvature, then curved space field theory is a more precise framework for the description even for the spatially flat case³. The ‘free’ version of the scalar field Lagrangian referred to here is the one coupled with scalar curvature. Accordingly, in this chapter the issue about dynamical equivalence of these two frames in curved space at the quantum level is addressed.

At the classical level the actions of the Jordan and Einstein frames are related by a conformal transformation of the metric and certain field reparameterisation of the scalar field. Concerning the reparameterisation of the metric, one always has the freedom to rescale the metric at the classical level. At the quantum level one may lose this freedom to rescale the spacetime because in the context of curved space QFT it is the particles produced from the curvature of spacetime (see, for example, [109]) that is an important property of curved space field theory. Even in the trivial conformal case - the conformally invariant field in the conformal flat spacetime - this is true. The reason is, because the theories at different energy scales are related by different renormalised coupling constants, as determined by the renormalisation group. The conformal anomaly that breaks the symmetry of the conformal transformation is proportional to the beta function [110]. Namely rescaling the spacetime effectively changes the energy scale and some massive particles are produced from the transformation. Also, in gravity the ground state energy

²Note that the term ‘semiclassical’ should not be confused with the same term which appears in next chapter where semiclassical function refers to the solution of Hamilton-Jacobi equation, which is in analogous to the WKB approximation and takes the lowest order of \hbar in the expansion.

³The description in flat space cannot depict some important features of curved space field theory such as the expectation value which is quadratic in the scalar fields being divergent and this can not be regularised by normal ordering as in flat space. This plays an important role in calculating the expectation value of the stress tensor. Of course our result can fit into flat space and no complex regularisation is required.

can not be assigned arbitrarily because it relates to spacetime curvature; thus the ‘new’ masses have significance. One needs to be more cautious with the transformation. The term conformal anomaly comes from the fact that the conformally invariant actions at the classical level lose their symmetry from quantum corrections and this is related to the non-zero trace of the stress tensor; thus it is also called the trace anomaly. The other famous anomalies in field theories are the chiral anomaly (see for example [111, 112]), and the gravitational anomalies (see for example [113, 114]). The trace anomaly has been widely studied in the context of curved spacetime [115–117]. Here, where we are transforming from the Jordan frame to the Einstein frame it is actually more subtle as they are not conformally invariant in the classical action but have only field reparameterisation invariance. Therefore the appropriate method is to study this topic by field redefinition/reparameterisation and this can actually include a larger class of transformations wherein the conformal anomaly is a special case⁴. Thus one needs to study this in terms of the field reparameterisation/ redefinition viewpoint instead of the conventional conformal anomaly technique.

To address this topic at the quantum level in the more general context of field redefinition/ reparameterisation, the discussion will be at the functional integral (path integral) level and later on even in the canonical quantisation level. In this way, before perturbation theory applies, we have a consistent understanding for the entire properties of the transformation. Notably it is not suitable to study the perturbative perspectives - comparing the scattering amplitude ‘order by order’ in both frames. This is because in the curved space their vacua are not identical meaning that in curved space their Bogolubov coefficient is not zero. To be precise through the transformation one can not exactly match the transition amplitude accurately from one frame at a certain order in perturbation theory to the other frame at the same order in perturbation theory. Thus one may not be surprised to find that some people have indicated that loop calculations in these two frames are not equivalent [119, 120] (note that these are in the context of quantisation of gravity). Therefore, studying this issue in the context of functional integration is more appropriate and that is the perspective we will adopt. The partition functional Z or generating function W in the path integral approach is the central object from which to derive all the relevant physical quantities including amplitudes and the S matrix. Therefore studying this partition function provides a *sufficient* condition⁵ for the dynamical behaviour.

⁴Some transformations start from a conformally invariant action with an auxiliary field, such as [118], and have gauge fixed to the Jordan frame; this can also fit in our approach.

⁵The reverse (the necessary condition) shall be true generically, though that may be not always be

Our main finding is the following: The resulting partition function after the transformation receives a non-trivial Jacobian in the functional measure. This can be derived as the expectation value of the trace of the stress tensor and reformulated as a non-conserved current within a total derivative. This shares a similarity to the trace anomaly in the scalar field part of Fujikawa’s method [121–123] if a specific symmetry is imposed, but one should not confuse this with the case we discuss which has no exact conformal symmetry. Our derivation, as functional method, is applicable for more generic field redefinition and can be applied beyond trace anomaly situation.

On the other hand, apart from the Jacobian as a non-conserved current, the rest are non-trivial boundary terms which cannot be set to zero in some cases in curved space. This can be understood as the conserved current coming from field reparameterisation, or the quantum field equation. This conserved current will be modified by a non-conserved Jacobian current obtained before. As mentioned, we study at the functional integral level, and specifically use the Schwinger-Dyson equations. The main feature is consistently expressed in terms of a “current” in which the constituents are the conserved current and the non-conserved current. After this the situation can be studied further, the conserved current can in fact be interpreted as boundary terms and the non-conserved Jacobian current can also be interpreted as a non-trivial boundary term, which means in the curved space field theory there exists non-trivial boundary terms which need to be taken into account when making a comparison.

We emphasise that the transformation from one frame to another only implies a field redefinition of the scalar field and the metric tensor. In a path integral formulation of quantum field theory (see e.g. [124] for a nice introduction), fields are dummy variables which are summed over. As long as the field redefinition does not violate any of the symmetries of the model, physics cannot be affected and physics cannot depend on the frame. For example, field redefinitions are an important part of the renormalisation program [18, 125].

In reviewing other developments in more geometrical approaches in the general field space, some interpretations propose that the non-conserved current term occurs in the measure when there is a non-trivial field redefinition where the complete set of modes does not respect the orthonormality condition in the general field space [126], which can recover Fujikawa’s results [121–123]. Therefore one might construct a geometrical object measure in such a way that it is invariant under coordinate redefinitions in the general field

guaranteed.

space [72], \mathcal{F} in which the field ϕ^i serves as coordinate points with its corresponding metric in that space G_{ij} in addition to the spacetime metric $g_{\mu\nu}$, in analogy to the non-linear sigma model $S = \frac{1}{2} \int dV g_{\mu\nu} G_{ij} \partial_\mu \phi^i \partial_\nu \phi^j$. In that space one has a connection defined by G_{ij} . This construction actually accommodates the Jacobian effect inside the newly defined functional measure in the new functional integral⁶. The definition of [72, 127, 128], with [129] as the original flat space version, realises the above statement. This is reminiscent of the treatment in quantum mechanics of ghost fields as in [130]. Their form is

$$Z = \int \prod_i d\sigma^i[\phi_\star; \phi] f[\phi_\star] \exp \frac{i}{\hbar} \{ S[\phi_\star; \sigma[\phi_\star; \phi]] - J_i \sigma^i[\phi_\star; \phi] \} \quad (3.1)$$

with $\sigma^i[\phi_\star; \phi] \equiv g^{ij}[\phi_\star] \frac{\delta}{\delta \phi_\star^j} \sigma[\phi_\star; \phi]$ being the tangent vector of the geodesic interval $\sigma[\phi_\star; \phi]$ connecting ϕ^i in the general field space to a fixed coordinate ϕ_\star^i , and

$$\prod_i d\sigma^i[\phi_\star; \phi] = | \det \sigma_{;j'}^i[\phi_\star; \phi] | \prod_i d\phi^i \quad (3.2)$$

$$= | g[\phi_\star] |^{-\frac{1}{2}} | g[\phi] |^{\frac{1}{2}} | \Delta[\phi_\star; \phi] | \prod_i d\phi^i \quad (3.3)$$

with $\Delta[\phi_\star; \phi]$ is the Van Vleck-Morette determinant. Our Jacobian related effect, resulting in changes of quantum field equations and the conserved current, comes from the fact that one needs to treat the general field space as a generally curved manifold instead of flat in this context. The ordinary definition of the partition function is not invariant in the general field space so that the Jacobian accounts for the inequivalence. Hence one may contend that the Jacobian related effect can be attributed as an illusory effect from the view point of the ordinarily defined non-geometric flat field space partition function to perform the field reparameterisation, in analogy to using a non-covariant approach to describe general relativity, which should not be an obstacle to the physical equivalence between frames.

Another observation is that one can add counter boundary terms artificially to offset the Jacobian effect; then we can retain the original quantum field equation and conserved current in the Schwinger-Dyson equation. This is in analogy to [131, 132], where they attempt to remove the anomaly by adding specific higher order terms apart from the normal Einstein-Hilbert action in L_{eff} . We can add certain terms - in our case they are some extra boundary terms - so as to cancel the Jacobian related effect. The freedom to add boundary terms to obtain the desired physical purpose is allowed since it does not influence the fields in the bulk, for example the Gibbons-Hawking term [133] in which they add a boundary term to obtain the correct field equation when in the open manifold.

⁶This chapter does not to address the quantum gravity issue; that means the gravitational fields are regarded as a general curved background classical field and not included in the path integral measure.

Through either approach mentioned above - using boundary terms to offset the non-conserved current, or constructing a new path integral with a newly defined measure to become field reparameterisation invariant - these two partition functions can be identical after either treatment is implemented. Therefore their n -point correlation function in the individual frame is deduced as identical up to conformal factors at the corresponding order. Thus maintaining physical equivalence.

This chapter is structured as follows: in Section 3.3 the classical action for non-minimally coupled scalar field theory and $F(R)$ theory in the Jordan frame is transformed to the Einstein frame with boundary terms preserved for the purpose of the following studies. We then present some brief background in Section 3.4 including the transformation of vacuum states, adiabatic expansion and importantly renormalisation of the expectation value of the stress tensor. The formulation at the quantum level in Section 3.5 is our main result; this section focuses on the quantum level of these two frames in the path integral description. There is a non-trivial Jacobian which can be reformulated as the expectation value of the stress tensor subject to renormalisation by the method introduced before. This is the important element one needs to take into account when making comparisons between frames. The importance of the boundary term and the Jacobian is revealed in the next section. The boundary term is actually a total derivative of some combination of fields taking their value on the boundary and can be reformulated as a conserved current with respect to the field reparameterisation. Also the non trivial Jacobian presents a non-conserved current and also can be reformulated in the sense of a non-trivial boundary term during reparameterisation. Putting these together in Section 3.6 in the framework of the Schwinger-Dyson equation of the currents and the quantum field equation one can find that this Jacobian current will modify the quantum field equation and the original conserved current. Combining these we propose a method to add an extra boundary term in order to offset these effects without affecting the field equation; in so far as the part that makes the different is also in a sense a boundary term we construct a counter boundary term in 3.6.1.

3.3 Classical Action

General Relativity is an extremely successful theory which has now been probed extensively. The beautifully simple Einstein-Hilbert action

$$S_{EH} = \int d^4x \sqrt{-g} \frac{R}{16\pi G} \quad (3.4)$$

incorporates all our current knowledge of gravity. In this equation, the symbol G stands for Newton's constant, g the determinant of the metric tensor $g_{\mu\nu}$ and R is the Ricci scalar which is uniquely determined by the metric tensor. However, in general, gravitational theories will contain higher dimensional terms such as R^2 or $R_{\mu\nu}R^{\mu\nu}$ and fields of different spins. For example when Einstein's gravity is coupled to the Standard Model one needs to introduce particles of spin 0, 1/2 and 1 on top of the spin two metric tensor which represents the graviton. In inflationary theories, one often introduces an inflaton which is represented by a scalar degree of freedom.

As we shall see shortly, the coupling of scalar fields allows for interesting complications as scalar fields can be coupled naturally in a non-minimal way to the gravitational field. With the discovery of a scalar boson at the CERN Large Hadron Collider we now know that there are such elementary scalar fields in nature. For example a neutral scalar field ϕ can be coupled to the Ricci scalar using $\phi^2 R$ which is a dimension four operator. Such a non-minimal coupling leads to an action of the type

$$S_{grav} = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} + \frac{1}{2} \xi \phi^2 R \right) \quad (3.5)$$

where ξ is the non-minimal coupling of the field ϕ to the curvature scalar.

As preparation for the following discussion, we shall first review the field redefinition used when transforming a gravitational theory from the Jordan to the Einstein frame at the classical level and then compare the theories at the semiclassical level, i.e., we shall not attempt to quantise gravity and will only consider quantum effects of the scalar field. The manifestation of boundary terms within total derivatives is also shown, which will be important in our study.

3.3.1 Transformation of the Action for a Non-minimally Coupled Scalar Theory

Before studying a general class of $F(R)$ theories, we shall review the case of a scalar field non-minimally coupled to the Ricci scalar. Note that we shall consider the transformation from the Jordan frame to the Einstein frame, but our results can trivially be used to consider the reversed transformation from the Einstein frame to the Jordan frame.

In the context of curved space quantum field theory, the usual 'free' scalar field theory is in the form of a non-minimally coupled scalar theory. The important difference is the additional terms $\frac{1}{2} \xi \phi^2 R$ which shows the scalar field coupling to the gravitational fields. Thus this is another reason, in addition to Section 3.1, to study such theories in curved space and so is the case of $F(R)$, in which the non-minimally coupled theory is a special

case. Note that the action in this case not only contains the curved space ‘free’ scalar field but it also contains the ordinary Einstein-Hilbert term. In the case of the non-minimal coupling the Jordan frame and Einstein frame are related by a field redefinition on $g_{\mu\nu}$ and ϕ respectively with the conformal factor ⁷

$$\Omega(x)^2 \equiv \exp[\sigma(x)] = 1 - 8\pi G\xi\phi(x)^2. \quad (3.6)$$

The coupling constant is chosen as $\xi = [4(n-1)]^{-1}(n-2) = \frac{1}{6}$ which is the conformal coupling in $n = 4$ dimensions, but our study is not restricted to the conformal case⁸. According to [104], the corresponding transformations for field, metric and all the relevant variables are⁹

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad (3.7)$$

$$\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu} \quad (3.8)$$

$$\sqrt{-\tilde{g}} = \Omega^n \sqrt{-g} = \Omega^4 \sqrt{-g} \quad (3.9)$$

$$d\tilde{\phi} = \frac{(1 - 8\pi G\xi(1 - 6\xi)\phi^2)^{\frac{1}{2}}}{1 - 8\pi G\xi\phi^2} d\phi \quad (3.10)$$

$$\tilde{V}(\tilde{\phi}) = \Omega^{-4} V(\phi) \quad (3.11)$$

$$\tilde{R} = \Omega^{-2} (R - 6\Box(\ln \Omega) - 6 \frac{g^{\mu\nu} \nabla_\mu \Omega \nabla_\nu \Omega}{\Omega^2}) \quad (3.12)$$

$$= \Omega^{-2} (R - 6 \frac{\Box \Omega}{\Omega}) \quad (3.13)$$

$$= \Omega^{-2} (R - \frac{12\Box\sqrt{\Omega}}{\sqrt{\Omega}} - 3 \frac{g^{\mu\nu} \nabla_\mu \Omega \nabla_\nu \Omega}{\Omega^2}). \quad (3.14)$$

This transformation¹⁰ is regarded as a generalised field reparameterisation and one always

⁷in some of the literature this transformation has a general appellation as a conformal transformation but it is a conformal transformation for the metric only. The form of the transformation for the scalar field is not exactly in the same form as a conformal transformation.

⁸The higher dimensional case is generic, though it is not our main concern,

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \Omega^2 g_{\mu\nu} \\ \det \tilde{g}_{\mu\nu} &= \Omega^{2n} \det g_{\mu\nu} \\ \tilde{\phi} &= \Omega^{\frac{1-n}{2}} \phi \\ \xi &= \frac{n-2}{4n-4} \end{aligned}$$

where n stands for the dimension of interest.

⁹The potential term here is for the completion for the formalism; the following discussion focus mainly on the free theory $V(\phi) = 0$. Also the potential term does not contribute to the stress tensor and thus Jacobian because stress tensor comes from the functional differential with respect to metric. In addition, the transformation is only singular at $\phi = (8\pi G\xi)^{-\frac{1}{2}}$ resulting in no gravitation degree of freedom and $\phi = 0$ it is the case of identical transformation. These two special cases are not of our interests.

¹⁰The (3.12) is the transformation of scalar curvature in four dimension and it can be simplified to

has the freedom to do so in a classical field theory. The Jordan Frame action is given as

$$S_J = \int d^4x \sqrt{-g} \left[\left(\frac{1}{16\pi G} - \frac{1}{2} \xi \phi^2 \right) R + \frac{1}{2} g_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - V(\phi) \right] \quad (3.15)$$

and this can be rewritten in the operator form with all the differential operators in between the scalar fields and we keep the boundary terms as they may not be zero in the curved space field theory - a total derivative term within the volume integral can always refer to a surface integral on boundary.

$$\begin{aligned} S_J &= \int d^4x \sqrt{-g} \left[\left(\frac{1}{16\pi G} - \frac{1}{2} \xi \phi^2 \right) R + \frac{1}{2} g_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - V(\phi) \right] \\ &= \int d^4x \sqrt{-g} \left[\left(-\frac{1}{2} \right) \phi (\square + \xi R) \phi - V(\phi) + \frac{R}{16\pi G} + \frac{1}{2} \nabla_\mu (g^{\mu\nu} \phi \nabla_\nu \phi) \right] \end{aligned} \quad (3.16)$$

The Jordan Frame action is then identical to the Einstein Frame action after the all the relevant reparameterisations (transformations) with (3.7), (3.9), (3.10), (3.11), (3.12), we obtain

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{16\pi G} \tilde{R} + \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}^\nu \tilde{\phi} - \tilde{V}(\tilde{\phi}) \right] \quad (3.17)$$

To be precise they are equivalent up to some boundary terms appearing during the transformation which may be neglected naively in the usual treatment in the flat space field theory. However we preserve them for they will be significant later on. Therefore we can write the boundary terms inclusively,

$$\begin{aligned} S_{E+boundary} &= \int d^4x \left\{ \left[-\frac{1}{2} \tilde{\phi} \square \tilde{\phi} + \tilde{V}(\tilde{\phi}) \right] + \frac{c^3 \tilde{R}}{16\pi G} \right. \\ &\quad \left. + \left[\frac{1}{2} \nabla_\mu (\tilde{g}^{\mu\nu} \tilde{\phi} \nabla_\nu \tilde{\phi}) - \frac{3 \square \ln \Omega}{8\pi G} \right] \right\} \end{aligned} \quad (3.18)$$

The last two terms are the ones that will take values at the boundary and for simplicity these term are symbolically defined as **boundary terms** in the Lagrangian and **surface terms** in the action.

$$\begin{aligned} &\int d^4x \frac{1}{2} \nabla_\mu (\tilde{g}^{\mu\nu} \tilde{\phi} \nabla_\nu \tilde{\phi}) - \frac{3 \Omega^2 \square \ln \Omega}{8\pi G \Omega^2} \\ &= \int d^4x \nabla_\mu \left[\frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\phi} \nabla_\nu \tilde{\phi} - \frac{3 \tilde{g}^{\mu\nu} \nabla_\nu \ln \Omega}{8\pi G} \right] \\ &= \int d\sigma_\mu \frac{1}{2} \left[\tilde{g}^{\mu\nu} \tilde{\phi} \nabla_\nu \tilde{\phi} - \frac{3 \tilde{g}^{\mu\nu} \nabla_\nu \ln \Omega}{4\pi G} \right] |_{\partial} \\ &\equiv (\mathbf{surface terms}), \end{aligned} \quad (3.19)$$

(3.13) by acting the Laplacian operator onto $\ln \Omega$ and performing partial integration within the action which integral over spacetime. The (3.14) can be checked to be equal to (3.13) by acting Laplacian operator onto $\sqrt{\Omega}$ and performing partial integration. We use (3.13) for most of calculation since these three are equivalent.

where $d\sigma_\mu$ is the 3-dimensional volume element. Note that the covariant derivatives can be replaced by ordinary derivatives as we are dealing with scalar fields. Therefore one can write that the action in the two frames is identical up to some boundary terms.

$$S_J = S_E + (\text{surface terms}) \quad (3.20)$$

and

$$\mathcal{L}_J = \mathcal{L}_E + \partial \cdot (\text{boundary terms}) \quad (3.21)$$

with the understanding that

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left(\frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{\phi} \tilde{\square} \tilde{\phi} - \tilde{V}(\tilde{\phi}) \right). \quad (3.22)$$

Note that we can start in the Jordan frame with non trivial boundary conditions such as the Gibbons-Hawking terms if an open space is considered without any complication.

3.3.2 Transformation of the Action for F(R) Scalar-tensor Gravitational Theories

In the following we consider the mapping of a $F(R) = f(\phi)R - V(\phi)$ theory, i.e. in the Jordan frame, to the Einstein frame. These models represent a subset of the ordinary $F(R)$ gravity models. This case is the generalisation of the previous one. If we take $f(\phi) = \phi^2$ we recover the results obtained for the non-minimally coupled scalar field.

The conformal factor is

$$\Omega(x)^2 \equiv 16\pi G \left| \frac{\partial F(x)}{\partial R} \right| \quad (3.23)$$

and provides the following redefinition of fields and metric in the same way as (3.7), (3.9), (3.11) and (3.13) but with the scalar field transformation in the following way,

$$\tilde{\phi} = \frac{1}{\sqrt{8\pi G}} \int \left\{ \frac{2f(\phi) + 6\left(\frac{df}{d\phi}\right)^2}{4f^2(\phi)} \right\}^{\frac{1}{2}} d\phi. \quad (3.24)$$

The the Jordan frame action

$$S_J := \int d^4x \sqrt{-g} [F(\phi, R) + \frac{1}{2} g_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi]$$

can be related to the Einstein Frame by $F(\phi, R) = f(\phi)R - V(\phi)$ and the above transformation relation.

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} + \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}^\nu \tilde{\phi} - U(\tilde{\phi}) \right] \quad (3.25)$$

where $U(\tilde{\phi}) = [16\pi G |f(\phi)|]^{-2}V(\phi)$. The resultant Einstein frame action with the boundary terms all together is

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[-\frac{1}{2} \tilde{\phi} \tilde{\Box} \tilde{\phi} - U(\tilde{\phi}) \right] + \frac{c^3 \tilde{R}}{16\pi G} + \frac{1}{2} \nabla_\mu (\tilde{g}^{\mu\nu} \tilde{\phi} \nabla_\nu \tilde{\phi}) + \frac{3\tilde{\Box} \ln \Omega}{8\pi G} \quad (3.26)$$

wherein the boundary terms have the same form as that of (3.18). This coincidence is because the form of the actions in both cases in the Einstein frame is identical apart from the potential terms and one can double check when $F(\phi, R) = f(\phi) R - V(\phi)$, then $\Omega^2 = 16\pi G f(\phi)$, the potential $U(\tilde{\phi}) = [16\pi G |f(\phi)|]^{-2}V(\phi)$ is equal to $\Omega^{-4}V(\phi)$, which is exactly $\tilde{V}(\tilde{\phi})$. The last two terms lead to the boundary terms:

$$\begin{aligned} & \int d^4x \frac{1}{2} \nabla_\mu (\tilde{g}^{\mu\nu} \tilde{\phi} \nabla_\nu \tilde{\phi}) + \frac{3\tilde{\Box} \ln \Omega}{8\pi G} \\ &= \int d^4x \nabla_\mu \left[\frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\phi} \nabla_\nu \tilde{\phi} - \frac{3\tilde{g}^{\mu\nu} \nabla_\nu \ln \Omega}{8\pi G} \right] \\ &= \int d\sigma_\mu \left[\frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\phi} \nabla_\nu \tilde{\phi} - \frac{3\tilde{g}^{\mu\nu} \nabla_\nu \ln \Omega}{8\pi G} \right] |_\partial \\ &\equiv (\text{surface terms}). \end{aligned} \quad (3.27)$$

Thus the same form as (3.20) and (3.21) will also apply.

Before we proceed to study the quantum level, some background knowledge about curved space field theory will be helpful.

3.4 Review of Curved Space Quantum Field Theory

3.4.1 General Aspects of Curved Space Quantum Field Theory

For completeness, we give a pedagogic review for the following subsections to give a brief introduction based on [68, 72] about some important features of the curved space quantum field theory. Following the explanation of the general features of quantum field theory in curved space [68], we understand that the vacuum state in the Minkowski space is uniquely defined whilst that in curved space does not work the same way [134]. The Poincaré group is not the symmetry group any more [135] as there is no Killing vector which leaves the line element invariant or preserves the positive frequency modes $u_k = \frac{1}{\sqrt{2\omega(2\pi)^3}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$. Even in certain special cases there may exist some coordinates analogous to rectangular coordinates in Minkowski space, but the physical properties are not the same as in flat space. By virtue of general covariance there are no privileged coordinates which enjoy this feature. This means some physical quantities can be well defined in one coordinate system

but the same quantity defined in other coordinates can be also equally valid; none of them shall be preferred. This causes ambiguity. The number of particles is a classic example.

Consider a scalar field ϕ , it can be expanded in terms of creation and annihilation operators

$$\phi(x) = a_i u_i + a_i^\dagger u_i^*, \quad (3.28)$$

it can also be expanded in another set of complete and orthonormal modes,

$$\phi(x) = \bar{a}_j \bar{u}_j + \bar{a}_j^\dagger \bar{u}_j^*. \quad (3.29)$$

Both of them are ‘equally good’ in the sense of general covariance, and their relations are, with the summation understood for repeated indices,

$$\bar{u}_j = \alpha_{ji} u_i + \beta_{ji} u_i^*, \quad (3.30)$$

or conversely,

$$u_i = \alpha_{ji}^* \bar{u}_j - \beta_{ji}^* \bar{u}_j^* \quad (3.31)$$

This is the well known Bogolubov transformation [136] where the scalar products $\alpha_{ij} = (\bar{u}_i, u_j)$ and $\beta_{ij} = -(\bar{u}_i, u_j^*)$ ¹¹ are the Bogolubov coefficients. Also the annihilation operator can be expressed as

$$a_i = \alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger \quad (3.32)$$

and

$$\bar{a}_j = \alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger. \quad (3.33)$$

An intriguing result is found when applying the annihilation operator from first formula to the vacuum of another set,

$$a_i | \bar{0} \rangle = \beta_{ji}^* | \bar{1}_j \rangle \neq 0,$$

in contrast to the fact that the annihilation operator acts on the vacuum state to give zero. It is more clear by use of the number operator $N_i = a_i^\dagger a_i$. The number of particles in the state of $|0\rangle$ in the view point of the u_i mode is

$$\langle \bar{0} | N_i | \bar{0} \rangle = |\beta_{ji}|^2. \quad (3.34)$$

That states the vacuum of the \bar{u}_j mode contains the $|\beta_{ji}|^2$ particles in the u_i mode. Also if $\beta_{ji} \neq 0$, the \bar{u}_i mode will always be the combination containing positive u_j and negative u_j^* frequency modes, due to the fact that there is no Killing vector to define positive

¹¹The scalar product is defined as $(\phi_1, \phi_2) = -i \int_t \phi_1 \overleftrightarrow{\partial}_t \phi_2 d^3x$

frequency modes $\mathcal{L}_\xi u_j = -i\omega u_j$ with $\omega > 0$ as can be done in flat space. Therefore β_{ij} reveals the general feature in curved space that the particle number does not have universal significance since the vacuum is not uniquely defined but observer dependent. This reveals that spacetime curvature can create particles due to a gravitational field; thus one can expect the renormalisation of the vacuum expectation values of physical quantities is also quite different from that in the flat space. The particle generation effect was discussed as early as 1939 by E. Schrödinger and in 1953 by B. S. DeWitt according to [68] and in [137–139] in terms of the concept of a particle detector [39, 140], where the detector response function for a late time observer per unit time in the energy E in n dimensions is,

$$\frac{\mathfrak{F}(E)}{T} = \frac{2^{2-n}\pi^{\frac{3-n}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^\infty dk \frac{k^{n-2}}{(k^2 + m^2)^{\frac{1}{2}}} |\beta_k|^2 \delta(E - (k^2 + m^2)^{\frac{1}{2}}) \quad (3.35)$$

$$= \frac{2^{2-n}\pi^{\frac{3-n}{2}}}{\Gamma(\frac{n-1}{2})} (E^2 - m^2)^{\frac{n-3}{2}} |\beta_{(E^2-m^2)^{\frac{1}{2}}}|^2 \theta(E - m), \quad (3.36)$$

where $\theta(E - m)$ is the Heaviside step function. The transformation of the basis mode can be done as the usual quantum state by inserting a complete set; the vacuum to many particle transition amplitude in different sets of modes in the curved space can be found [70],

$$\langle \bar{0} | 1_{j_1}, 1_{j_2}, \dots, 1_{j_k} \rangle = \begin{cases} i^{\frac{k}{2}} \langle \bar{0} | 0 \rangle \sum_\rho \Lambda_{\rho_1 \rho_2} \cdots \Lambda_{\rho_{k-1} \rho_k}, & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \quad (3.37)$$

$$\langle \bar{1}_{j_1}, \bar{1}_{j_2}, \dots, \bar{1}_{j_k} | 0 \rangle = \begin{cases} i^{\frac{k}{2}} \langle \bar{0} | 0 \rangle \sum_\rho V_{\rho_1 \rho_2} \cdots V_{\rho_{k-1} \rho_k}, & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \quad (3.38)$$

where ρ is all the distinct permutations of $\{j_1, \dots, j_k\}$ and

$$\begin{cases} \Lambda_{ij} &= -i\beta_{kj}\alpha_{ik}^{-1} \\ V_{ij} &= i\beta_{jk}^*\alpha_{ki}^{-1}. \end{cases} \quad (3.39)$$

Therefore we can understand that the number of particles is indeed an observer dependent quantity in curved space. More importantly there is no preferred choice of vacuum, which makes the calculation of the expectation values of physical quantities unclear. We need to employ a specific method to deal with this situation in curved space.

3.4.2 DeWitt Schwinger Representation of Green's Function

As mentioned above, the troublesome situation in curved space needs to be tackled. In this subsection we will introduce a specific method to allow physical quantities to have an

expansion. The expansion is in terms of slowness of change of spacetime; therefore one can have a controllable manner in which to include the accuracy of spacetime change.

In terms of an expansion it is also called an adiabatic expansion to the n -th order; this is because each order increases the derivatives and thus inverse powers of time unit, which corresponds to increasing slowness of the change of physical quantities, such as particle production or the stress tensor, by the spacetime curvature order by order. This expansion is a tool to deal with the situation in curved space because as mentioned earlier in 3.4.1 one cannot specify a preferred set uniquely. However as stated by [141], any physical state can be specified by a complete set of eigenvalues with mutual commuting observables. The goal is to construct a certain expansion, though it may not be unique, which can reproduce any physical expectation value and importantly can increase the accuracy of slowness in a controlled way; thus controlling the amount of deviation from flat space.

The adiabatic expansion works in a similar but not exactly the same way as the perturbation methods commonly used in physics. The adiabatic expansion includes more and more slowly changing modes to the physical quantity, while the perturbation is to add more and more drastically changed factors into the original quantity. Regarding the perturbation method, there are some common examples. In particle physics, which includes higher energy corrections deviating from classical physics, this is in terms of the order of the coupling constant. In cosmology the scalar field is written as $\phi = \phi_0 + \delta\phi$ where the ϕ_0 is the classical field and $\delta\phi$ is the quantum fluctuation. In a weak gravitational field [79, 142, 143], $g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}$ contains a fixed flat background and a perturbation $f_{\mu\nu}$, which can account for the graviton. Similarly in the background method [144, 145] for arbitrary background, the fluctuation is to be added to the background metric as a perturbation. The only common point is to increase the accuracy of the descriptions. While they all include higher order derivatives as perturbations to increase the accuracy of the deviation from a certain background reference which is important at high energies or small scales; yet the purpose of the adiabatic expansion is to increase the accuracy depicting the changing of spacetime. The lower adiabatic order represents the major changes and the higher adiabatic order are more subtle changes; therefore for a slowly expanding universe adiabatic order 2 is enough and for a quickly changing universe it may require adiabatic order 4.

We now follow the treatment of [146] and [147, 148] for the Riemann normal coordinate analysis¹². The Riemann normal coordinate is a coordinate system in which an origin point

¹²The sign difference is due to the convention.

Q has a unique geodesic connecting to any point of its neighbourhood, called a normal neighbourhood of Q . An arbitrary point P in this region can be expressed in coordinates,

$$y^\mu = \lambda \xi^\mu, \quad (3.40)$$

where λ is an affine parameter with $\lambda = 0$ at point Q and ξ^μ is the tangent vector to the geodesic at Q , $\xi^\mu = \frac{dx^\mu}{d\lambda} \big|_Q$. Moreover along any geodesic through Q , ξ^μ is constant and independent of λ ; therefore the geodesic equation is $\frac{d^2 y^\alpha}{d\lambda^2} = 0$, which means

$$\Gamma^\alpha_{\beta\gamma}(y) \frac{dy^\beta}{d\lambda} \frac{dy^\gamma}{d\lambda} = \Gamma^\alpha_{\beta\gamma}(y) \xi^\beta(y) \xi^\gamma(y) = 0 \quad (3.41)$$

where $\xi^\beta(y)$ is a certain tangent vector at Q along a geodesic. Multiplying λ^2 , it becomes $\Gamma^\alpha_{\beta\gamma}(y) y^\beta y^\gamma = 0$ and at Q we have $\Gamma^\alpha_{\beta\gamma}(Q) y^\beta y^\gamma = 0$. Since ξ^β directs any geodesic through Q , it means

$$\Gamma^\alpha_{\beta\gamma}(Q) = 0. \quad (3.42)$$

We are also allowed to locally diagonalise the metric at Q so that $g_{\mu\nu}(Q) = \eta_{\mu\nu}$. We can expand the connection around Q at $y^\alpha = 0^{13}$,

$$\Gamma^\alpha_{\beta\gamma}(Q) = \Gamma^\alpha_{\beta\gamma}(0) + [\partial_\mu \Gamma^\alpha_{\beta\gamma}(0)] y^\mu + \frac{1}{2!} [\partial_\mu \partial_\nu \Gamma^\alpha_{\beta\gamma}(0)] y^\mu y^\nu + \dots \quad (3.43)$$

By the same token with the result from [149, 150], the metric can be expanded,

$$\begin{aligned} g_{\mu\nu}(y) = & \eta_{\mu\nu} + \frac{1}{3} R_{\alpha\mu\beta\lambda}(0) y^\mu y^\lambda - \frac{1}{3!} \nabla_\mu R_{\alpha\gamma\beta\lambda}(0) y^\lambda y^\mu y^\gamma \\ & + \frac{1}{5!} (6 \nabla_\mu \nabla_\lambda R_{\alpha\delta\beta\gamma}(0) + \frac{16}{3} R_{\lambda\beta\mu}{}^\rho(0) R_{\gamma\alpha\delta\rho}(0)) y^\lambda y^\mu y^\gamma y^\delta. \end{aligned} \quad (3.44)$$

In curved space the correct tensor density adjustment to maintain covariance requires the following difference from flat space for the volume element d^4x , the Dirac delta function $\delta(x, y)$ and the propagator $D_{ij}(x, x') = [\delta_{ij} \square_x + m_{ij}^2(x)] \delta(x, x')$ respectively,

$$\sqrt{-g} d^4x \quad (3.45)$$

$$\frac{1}{\sqrt{-g}} \delta(x, y) \quad (3.46)$$

$$[\square + m^2 + \xi R] \frac{\delta(x, y)}{\sqrt{-g}}. \quad (3.47)$$

Defining the densitised Green's function,

$$\mathcal{G}(x, x') = (-g)^{\frac{1}{4}} G(x, x'), \quad (3.48)$$

¹³The condition for a Riemann coordinate is $\Gamma^\alpha_{\beta\gamma}(0) = 0$, $\partial_{(\mu} \Gamma^\alpha_{\beta\gamma)}(0) = 0$ and $\partial_{(\mu} \partial_{\nu} \Gamma^\alpha_{\beta\gamma)}(0) = 0$. In this subsection only, the n indices within the parenthesis have all possible permutations and also are divided by $\frac{1}{n!}$.

its expansion to the fourth order of derivatives is,

$$\begin{aligned} \mathcal{G}(x, x') \approx & \int \frac{d^4 k}{(2\pi)^4} e^{-iky} [a_0(x, x') + a_1(x, x')(-\frac{\partial}{\partial m^2}) \\ & + a_2(x, x')(\frac{\partial}{\partial m^2})^2 + \dots] \frac{1}{k^2 - (m^2 - i\varepsilon)}, \end{aligned} \quad (3.49)$$

where k is the momentum, and

$$a_0(x, x') \equiv 1 \quad (3.50)$$

$$a_1(x, x') = (\frac{1}{6} - \xi)R - \frac{1}{2}(\frac{1}{6} - \xi)(\nabla_\alpha R)y^\alpha - \frac{1}{3}a_{\alpha\beta}y^\alpha y^\beta \quad (3.51)$$

$$a_2(x, x') = \frac{1}{2}(\frac{1}{6} - \xi)^2 R^2 + \frac{1}{3}a_\lambda^\lambda, \quad (3.52)$$

with

$$\begin{aligned} a_{\alpha\beta} = & \frac{1}{2}(\xi - \frac{1}{6})\nabla_\beta \nabla_\alpha R + \frac{1}{120}\nabla_\beta \nabla_\alpha R - \frac{1}{40}\nabla^\lambda \nabla_\lambda R_{\alpha\beta} \\ & - \frac{1}{30}R_\alpha^\lambda R_{\lambda\beta} + \frac{1}{60}R_{\alpha\beta}^\kappa R_{\kappa\lambda} + \frac{1}{60}R_{\alpha}^{\lambda\mu\kappa} R_{\lambda\mu\kappa\beta}. \end{aligned}$$

With the integration to represent the propagator, and a small negative imaginary part included to give the correct integration contour,

$$\frac{1}{k^2 - (m^2 - i\varepsilon)} = -i \int_0^\infty ds e^{is(k^2 - m^2 + i\varepsilon)}, \quad (3.53)$$

the expansion of the densitised Green's function in n dimensions (3.49) can be expressed as follows in analogy to (2.29) but with a different context,

$$\mathcal{G}(x, x') = -i(4\pi)^{-\frac{n}{2}} \int_0^\infty i ds (is)^{-\frac{n}{2}} \exp[-im^2 s + (\frac{\sigma}{2is})] F(x, x'; is) \quad (3.54)$$

where $\sigma(x, x') = \frac{1}{2}y_\alpha y^\alpha$ is one half of the square of proper distance between x and x' and $F(x, x'; is)$ contains the expansion part,

$$F(x, x'; is) \approx a_0(x, x') + a_1(x, x')is + a_2(x, x')(is)^2 + \dots = \sum_{j=0}^\infty a_j(x, x')(is)^j. \quad (3.55)$$

Combining (3.48) and (3.54) we have the DeWitt Schwinger proper time representation [69–71, 151, 152] for n dimensions,

$$G(x, x') = -\Delta^{\frac{1}{2}}(x, x')(4\pi)^{-\frac{n}{2}} \int_0^\infty i ds (is)^{-\frac{n}{2}} \exp[-im^2 s + (\frac{\sigma}{2is})] F(x, x'; is) \quad (3.56)$$

where $\Delta = -\det[\partial_\mu \partial_\nu \sigma(x, x')][g(x)g(x')]^{-\frac{1}{2}}$ is the Van Vleck determinant [153]; it can be simplified as $\frac{1}{\sqrt{-g(x)}}$ when we use Riemann normal coordinates around point. The above Green's function is the exact form; if (3.55) is substituted into the Green's function, the expansion can be obtained,

$$G(x, x') \approx \frac{-i\pi\Delta^{\frac{1}{2}}(x, x')}{(4\pi i)^{\frac{n}{2}}} \sum_{j=0}^\infty a_j(x, x')(-\frac{\partial}{\partial m^2})^j [(\frac{2m^2}{-\sigma})^{\frac{n-2}{4}} H_{\frac{n-2}{2}}^{(2)}((2m^2\sigma)^{\frac{1}{2}})], \quad (3.57)$$

where $H_a^{(2)}$ is the Hankel function of the second kind.

Once the form of the expansion is understood we can move on to calculate a physical quantity - the expectation value of stress tensor $T_{\mu\nu}$.

3.4.3 General Aspects of Renormalisation of Stress Tensors

Since the meaning of particle number is vague it is more objective to probe physical quantities, such as $\langle \psi | T_{\mu\nu}(x) | \psi \rangle$, which is defined locally and can be related to other observers by the normal tensor transformation. The locally defined quantities are not like the particle number which is influenced by field modes and defined globally. The stress tensor is the source of gravity in Einstein's field equations and its expectation value under the curved space classical background with the quantisation of matter fields requires specific techniques to regularise its infinities as we will see. Any expectation value quadratic in the fields will occur, unlike the divergences in Minkowski space which can be discarded by normal ordering. In flat space we can always rescale the zero point energy in order to take off the infinite value, the only meaningful quantity is the energy difference. But in the context when gravity is taken into account, mass-energy itself is the source of gravitation and curvature, which cannot be set arbitrarily. In this subsection, we provide a pedagogic review of the general methods used to deal with divergences in curved space.

Following the discussion of regularisation in curved space quantum field theory in [154] and mainly in [68], Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}, \quad (3.58)$$

will be treated semiclassically with the matter field quantised and gravitational field kept classical,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda_B g_{\mu\nu} = -8\pi G_B \langle T_{\mu\nu} \rangle, \quad (3.59)$$

with the B subscript representing the bare quantity. The classical action is divided into two parts

$$S = S_{\text{gravity}} + S_{\text{matter}}; \quad (3.60)$$

the variation of $\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{gravity}}}{\delta g^{\mu\nu}} = 0$ yields the left hand side of Einstein's equation while the variation of $\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} = T_{\mu\nu}$ yields the right hand side of the equation. Moreover, the variation of the generating function W_{matter} is on the right hand side of (3.59). In reference to (2.15), one obtains

$$\frac{2}{\sqrt{-g}} \frac{\delta W_{\text{matter}}}{\delta g^{\mu\nu}} = \frac{\langle \text{out}, 0 | T_{\mu\nu} | \text{in}, 0 \rangle}{\langle \text{out}, 0 | \text{in}, 0 \rangle} = \langle T_{\mu\nu} \rangle. \quad (3.61)$$

With some help from the variational formulae,

$$\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} g_{\rho\sigma} \quad (3.62)$$

$$\delta\sqrt{-g} = \frac{1}{2}(\sqrt{-g})g^{\mu\nu}\delta g_{\mu\nu} \quad (3.63)$$

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + g^{\rho\sigma}g^{\mu\nu}(\delta\nabla_\nu\nabla_\mu g_{\rho\sigma} + \delta\nabla_\nu\nabla_\sigma g_{\rho\mu}) \quad (3.64)$$

the classical stress tensor for a non-minimally coupled ($\xi \neq 0$) massive scalar field in n dimensions is,

$$\begin{aligned} T_{\mu\nu} = & (1 - 2\xi)\nabla_\mu\phi\nabla_\nu\phi + (2\xi - \frac{1}{2})g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi \\ & - 2\xi\phi\nabla_\mu\nabla_\nu\phi + \frac{2}{n}\xi g_{\mu\nu}\phi\Box\phi \\ & - \xi(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{2(n-1)}{n}\xi Rg_{\mu\nu})\phi^2 \\ & 2(\frac{1}{4} - (1 - \frac{1}{n})\xi)m^2g_{\mu\nu}\phi^2. \end{aligned} \quad (3.65)$$

Before go further, a specific expansion scheme for the generating function needs to be applied in order to properly deal with regularisation in curved space. This expansion will apply to the Green's function, thus the generating function. It is useful for calculating finite values for the stress tensor. First, invoking some previous formulas (2.23),

$$W = -i\hbar \ln Z[0] \quad (3.66)$$

$$= -\frac{i\hbar}{2}\text{Tr}[\ln(-G_{ij})]. \quad (3.67)$$

At short distance (high energy), the Green's function is divergent; therefore using it as a tool to calculate the stress tensor as we will do later we also need to first utilise a regularisation of the Green's function. Once it has been regularised the same process can be passed to the calculation of the stress tensor. To renormalise, we adopt dimensional regularisation, in which we allow the dimensionality to analytically continue away from four; thus the divergent part can be dealt with in a controlled way.

With the expansion representation (3.56), the generating function W can be expressed as (also known as the one loop effective action),

$$W = \frac{1}{2}i \int_{m^2}^{\infty} dm^2 \int d^n x \sqrt{-g} G(x, x) \quad (3.68)$$

$$= \int \sqrt{-g} L_{\text{eff}}(x) d^n x \quad (3.69)$$

with

$$L_{\text{eff}} = \frac{1}{2}i \lim_{x \rightarrow x'} \int_{m^2}^{\infty} dm^2 G(x, x'); \quad (3.70)$$

in terms of the adiabatic expansion of the Green's function (3.57), it is

$$L_{\text{eff}} \approx \lim_{x \rightarrow x'} \frac{\Delta^{\frac{1}{2}}(x, x')}{2(4\pi)^{\frac{n}{2}}} \sum_{j=0}^{\infty} a_j(x, x') \int_0^{\infty} (is)^{j-1-\frac{n}{2}} e^{-i(m^2 s - \frac{\sigma}{2s})} i ds \quad (3.71)$$

$$= \lim_{x \rightarrow x'} \frac{1}{2} (4\pi)^{-\frac{n}{2}} \sum_{j=0}^{\infty} a_j(x, x') \int_0^{\infty} (is)^{j-1-\frac{n}{2}} e^{-im^2 s} i ds \quad (3.72)$$

$$= \frac{1}{2} (4\pi)^{-\frac{n}{2}} \sum_{j=0}^{\infty} a_j(x) (m^2)^{\frac{n}{2}-j} \Gamma(j - \frac{n}{2}), \quad (3.73)$$

where n is analytically continued to the complex plane and $a_j(x) = a_j(x, x')$ in the limit $x \rightarrow x'$. Inserting a reference energy scale μ to the above formula, which maintains the dimension of the Lagrangian to be $[L]^{-4}$,

$$L_{\text{eff}} \approx \frac{1}{2} (4\pi)^{-\frac{n}{2}} \left(\frac{m}{\mu}\right)^{n-4} \sum_{j=0}^{\infty} a_j(x) m^{4-2j} \Gamma(j - \frac{n}{2}). \quad (3.74)$$

There are only $\frac{n}{2} + 1$ terms that have ultraviolet divergences due to processing poles in the gamma function, whose divergence in $n \rightarrow 4$ is

$$\Gamma(-\frac{n}{2}) = \frac{4}{n(n-2)} \left(\frac{2}{4-n} - \gamma\right) + O(n-4) \quad (3.75)$$

$$\Gamma(1 - \frac{n}{2}) = \frac{2}{2-n} \left(\frac{2}{4-n} - \gamma\right) + O(n-4) \quad (3.76)$$

$$\Gamma(2 - \frac{n}{2}) = \frac{2}{4-n} - \gamma + O(n-4). \quad (3.77)$$

Notice that L_{eff} diverges at the lower end of the integration; it is because the $\frac{\sigma}{2s}$ damping factor vanishes when $\lim_{x \rightarrow x'}$ applies while the upper end is convergent because there is $m^2 - i\varepsilon$ implicitly included. To be precise, in reference to (3.57) the divergent part in four dimensions comes from the first three terms in (3.74), with (3.50), (3.51) and (3.52) provided:

$$L_{\text{div}} = - \lim_{x \rightarrow x'} \frac{\Delta^{\frac{1}{2}}(x, x')}{32\pi^2} \int_0^{\infty} \frac{ds}{s^3} e^{-i(m^2 s - \frac{\sigma}{2s})} [a_0(x, x') + a_1(x, x')is + a_2(x, x')(is)^2] \quad (3.78)$$

$$= - (4\pi)^{-\frac{n}{2}} \left\{ \frac{1}{n-4} + \frac{1}{2} [\gamma + \ln(\frac{m^2}{\mu^2})] \right\} \left(\frac{4m^2 a_0}{n(n-2)} - \frac{2m^2 a_1}{n-2} + a_2 \right). \quad (3.79)$$

All other terms higher than $a_3(x, x')$ are finite in four dimensions. This divergence will also appear when calculating $\langle T_{\mu\nu} \rangle$ as we will see. The divergent part is made up only by the local tensor, $R_{\mu\nu\rho\sigma}$ and its contraction. This is because the ultraviolet behaviour can only be probed by the local geometry and is not dependent on large scale quantities, such as topology.

The renormalised part of Lagrangian can be written as

$$L_{\text{ren}} \equiv L_{\text{eff}} - L_{\text{div}}; \quad (3.80)$$

therefore the renormalised part is an expansion with terms after $j = 3$ in four dimensions with $x = x'$. The (3.79) is obtained by partial integration three times. When L_{eff} is renormalised, W will become W_{ren} accordingly. These divergent terms can be reabsorbed in to the gravitational side of Einstein's field equation since they are all geometrical.

In terms of the semiclassical action, with (3.60) written as,

$$S = S_{\text{gravity}} + W_{\text{matter}}, \quad (3.81)$$

after renormalisation, it becomes

$$S = (S_{\text{gravity}})_{\text{ren}} + W_{\text{ren}}.$$

This is because the divergent part of W is reabsorbed into S_{gravity} having renormalised the coupling constants and thus W is finite. So is $\langle T_{\mu\nu} \rangle$, which is derived from W by (3.61). Accordingly the semiclassical Einstein's field equation (3.59) with renormalised coefficients will read

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda_{\text{ren}} g_{\mu\nu} + \alpha H_{\mu\nu}^{(1)} + \beta H_{\mu\nu}^{(2)} + \gamma H_{\mu\nu} = -8\pi G_{\text{ren}} \langle T_{\mu\nu} \rangle_{\text{ren}}, \quad (3.82)$$

where

$$\Lambda(\mu) \equiv \Lambda_{\text{B}} + A 8\pi G_{\text{B}}, \quad (3.83)$$

with $A = \frac{4m^4}{(4\pi)^{\frac{n}{2}} n(n-2)} \left\{ \frac{1}{n-4} + \frac{1}{2}[\gamma + \ln(\frac{m^2}{\mu^2})] \right\}$ and

$$G(\mu) = \frac{G_{\text{B}}}{1 + 16G_{\text{B}} B} \quad (3.84)$$

with $B = \frac{2m^2(\frac{1}{6}-\xi)}{(4\pi)^{\frac{n}{2}}(n-2)} \left\{ \frac{1}{n-4} + \frac{1}{2}[\gamma + \ln(\frac{m^2}{\mu^2})] \right\}^{14}$.

The rest of the new terms do not appear in the usual Einstein-Hilbert Lagrangian and they are regarded as higher order quantum corrections to general relativity in the renormalisation of matter fields. The renormalised action will take a different form if one includes the renormalisation of other fields. Alternatively, in terms of the EFT point of view [17–23] one can add these higher order terms and they are subject to observational

¹⁴In the case we considered here, the renormalisation group ‘running’ of G is in the context of the non-minimally coupled scalar field. In the same way the running of the cosmological constant is due to the presence of the matter fields.

bounds [155–157]. The form of these higher order correction terms are

$$\begin{aligned}
H_{\mu\nu}^{(1)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} R^2 d^n x \\
&= 2\nabla_\nu \nabla_\mu R - 2g_{\mu\nu} \square R - \frac{1}{2} g_{\mu\nu} R^2 + 2R R_{\mu\nu}
\end{aligned} \tag{3.85}$$

$$\begin{aligned}
H_{\mu\nu}^{(2)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} R^{\alpha\beta} R_{\alpha\beta} d^n x \\
&= \nabla_\nu \nabla_\mu R - \frac{1}{2} g_{\mu\nu} \square R - \square R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} + 2R^{\alpha\beta} R_{\alpha\beta\mu\nu} \\
&= 2\nabla_\alpha \nabla_\nu R_\mu^\alpha - \square R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \square R + 2R_\mu^\alpha R_{\alpha\nu} - \frac{1}{2} g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta}
\end{aligned} \tag{3.86}$$

$$\begin{aligned}
H_{\mu\nu} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} d^n x \\
&= -\frac{1}{2} g_{\mu\nu} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + 2R_{\alpha\beta\gamma\delta} R_\nu^{\alpha\beta\gamma} - 4\square R_{\mu\nu} + 2\nabla_\nu \nabla_\mu R \\
&\quad - 4R_{\mu\alpha} R_\nu^\alpha + 4R^{\alpha\beta} R_{\alpha\mu\beta\nu}.
\end{aligned} \tag{3.87}$$

Once L is renormalised, so is W . Equation (3.61) can be calculated as well. Thus the expansion scheme is actually passed down from the Green's function to the expectation value of the stress tensor so one can also calculate directly from the Green's function $G^{(1)}(x, x')$, Hadamard's elementary function $\langle 0 | \{\phi(x), \phi(x')\} | 0 \rangle$ which is quadratic in the scalar field, with the divergence up to a certain adiabatic order subtracted. Then one may apply all the differentiation, mass, etc to the Green's function to construct the form of $\langle T_{\mu\nu} \rangle$ because the stress tensor is composed by the combination of the quadratic of scalar field and its derivation and mass term.

3.5 Frame Transformation in the Quantum Level

In this section we consider the path integral quantisation formalism. The partition function for the Jordan frame theory is given by:

$$Z_J = N \int d\mu[\phi] \exp \left(\frac{i}{\hbar} \left(\int d^4 x \mathcal{L}_J + \int d^4 x \sqrt{-g} J_\phi \phi \right) \right). \tag{3.88}$$

We now show that it is equivalent to the partition function of the gravitational theory in the Einstein frame if the field redefinition is done properly. In order to compare, we shall actually work backwards and start from Z_E defined by the following with all the variables in the Einstein frame,

$$Z_E = \tilde{N} \int d\mu[\tilde{\phi}] \exp \left(\frac{i}{\hbar} \left(\int d^4 x \mathcal{L}_E + \int d^4 x \sqrt{-\tilde{g}} \tilde{J}_\phi \tilde{\phi} \right) \right) \tag{3.89}$$

where \tilde{N} is the normalisation. Performing the field redefinitions using the description defined above we obtain

$$Z_E = \tilde{N} \int \det C_{N'N} d\mu[\phi] \exp \frac{i}{\hbar} \left(\int d^4x (\mathcal{L}_J - \partial \cdot (\mathbf{boundary terms})) + \sqrt{-g} J_\phi \phi \right), \quad (3.90)$$

where $C_{N'N}$ is the Jacobian of the transformation of the measure of the path integral. It is defined by

$$d\mu[\tilde{\phi}_{N'}] = \det C_{N'N} d\mu[\phi_N], \quad (3.91)$$

where the N and N' represent the modes of the scalar fields in curved space. In the sequel, we will omit the indices. One can show that the Jacobian is proportional to the trace of the stress tensor. Note that the calculation is in the same form as the famous anomaly result [72, 121–123, 126] by the functional method, which possesses symmetry classically but the symmetry is broken at the quantum level; it is however conceptually very different from the anomaly calculation since we are not considering symmetry transformations but rather field redefinitions. Furthermore, our derivation is more generalised such that any field redefinition can be accommodated which is the purpose of our discussion. Namely, the anomaly can be regarded as a special case in this formalism when the form of the transformation is chosen in the corresponding way. The result can be applied in the wide class of scalar tensor theories. Also the deep reasoning of the anomaly is understood as the orthonormality of the complete set not being respected which will produce a non-trivial Jacobian [126]. The same reasoning applies for a general field redefinition. With the detail shown later in Section A, one obtains

$$i\hbar \ln(\det C) = -\frac{1}{2} \int dV \langle T^\mu_\mu \rangle_{\text{ren}} \quad (3.92)$$

with the expectation value calculated in the curved space in 3.4.3 with the help of the adiabatic expansion of the generating function $W = -i \ln Z[0]$ or Green's function. Also dV is understood as the covariant volume element $\sqrt{-g} d^4x$. The exact form of $\langle T^\mu_\mu \rangle_{\text{ren}}$, and thus the Jacobian, is indeed dependent entirely on the background geometry and also the exact form of $F(R)$ apart from the conformal anomaly case. This must be calculated in the corresponding model of interest. The calculation of the trace of the stress tensor is an important and complex subject which has been studied for a long time in curved space QFT. One can find many useful results which are based on the approach in 3.4.3. Here we only list some famous examples in the literature as referenced below, since our purpose in this chapter is not to conduct the calculation of the expectation value but to study the reasoning behind the equivalence issue between frames.

- If we take $\xi = \frac{1}{6}$ and a massless scalar field, there is a well known result for the conformal coupling case, which causes the famous conformal anomaly. The expectation value of the stress tensor is $\langle T_\mu^\mu \rangle_{\text{ren}} = \frac{2}{\sqrt{-g(x)}} g^{\mu\nu} \frac{\delta W}{\delta g^{\mu\nu}} = -\frac{\Omega}{\sqrt{-g(x)}} \frac{\delta W}{\delta \Omega}$. With certain regularisation schemes in curved space imposed, such as dimensional regularisation, the consequent expectation value of the stress tensor is $\langle T_\mu^\mu \rangle_{\text{ren}} = \frac{1}{4\pi^2} [\frac{1}{120} C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} - \frac{1}{360} G + \frac{1}{180} \square R]$ from the renormalisation of the stress tensor [68, 72, 117, 158–163] with $G \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2$ is the Gauss-Bonnet topological invariant and $C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{3} R^2$ is the square of Weyl tensor. This expectation value is purely local and only dependent on the geometry without dependence on the quantum state.

- We may also take the general case in which the coupling is not chosen to be $\frac{1}{6}$ and we may allow the existence of a mass term for the scalar field. The stress tensor is $T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = (1 - 2\xi) \nabla_\mu \phi \nabla_\nu \phi + (2\xi - \frac{1}{2}) g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi + \frac{1}{2} \xi g_{\mu\nu} \phi \square \phi - \xi [R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{3}{2} \xi R g_{\mu\nu}] \phi^2 - m^2 g_{\mu\nu} \phi^2$. Its expectation value can be evaluated with a certain regularisation scheme in curved space [164–167] $\langle T_{\mu\nu} \rangle_{\text{ren}} = (\frac{1}{64\pi^2} g_{\mu\nu}) (m^2 [m^2 + (\xi - \frac{1}{6}) R] [\Psi(\frac{3}{2} + \nu) + \Psi(\frac{3}{2} - \nu) - \ln(12m^2 R^{-1})] - m^2 (\xi - \frac{1}{6}) R - \frac{1}{18} m^2 R - \frac{1}{2} (\xi - \frac{1}{6})^2 R^2 + \frac{1}{2160} R^2)$ in which $\Psi(z) =: \frac{\Gamma'(z)}{\Gamma(z)}$. The contraction of indices can be worked out straightforwardly. This quantity is dependent not only on the geometry but also on the global (long distance) behaviour and the quantum state chosen. This is also subject to the background geometry chosen; for the case here de Sitter space is considered.

In another background spacetime geometry, which is not to be studied in this thesis, it is generally quite difficult to calculate functional differentiation of W_{ren} with respect to $g_{\mu\nu}$ to obtain $\langle T_{\mu\nu} \rangle_{\text{ren}}$. This is because one would need to know W_{ren} for all geometries $g_{\mu\nu}$. One would look for a more technical method, such as the point splitting method (see for example [168–171]) or ζ function regularisation (see for example [172–174]). We only cover the general guide lines in the following. First, one needs to solve the field equation to have a complete set and use it to construct the Green's function $G^{(1)}(x, x')$ and Hadamard's elementary function $\langle 0 | \{ \phi(x), \phi(x') \} | 0 \rangle$ since it is quadratic in the scalar field and can be used to calculate the stress tensor. Then drop the n -th adiabatic terms in the expansion to leave the renormalised Green's function. Act on this Green's function with all the differentiation, mass, etc to construct the form of $\langle T_{\mu\nu} \rangle$ because the stress tensor is composed by the combination of the quadratic of scalar field and its derivation and mass term. Finally take the

limit $x \rightarrow x'$.

- The $F(R)$ case is entirely dependent on the exact form of $f(\phi)$ and $V(\phi)$ within which the previous two results can be recovered when certain $f(\phi)$ is chosen and generally adapt the same features for the whole foregoing and following discussion—as will be shown the conserved current (as boundary terms) being modified by the Jacobian (as another current but non-conserved).

With the derivation of Section A for the Jacobian factor (3.92) and the boundary terms from Section 3.3, our main result is

$$Z_E = \tilde{N} \int d\mu[\phi] \exp \frac{i}{\hbar} \left(\int dV \left(\mathcal{L}_J + \frac{1}{2} \langle T_\mu^\mu \rangle_{\text{ren}} - \partial \cdot (\mathbf{boundary terms}) \right) + \sqrt{-g} J_\phi \phi \right), \quad (3.93)$$

With the help of the known result of $\langle T_\mu^\mu \rangle_{\text{ren}}$, the result is understandable. While the classical boundary term was known, the semiclassical correction is new and should be taken into account when performing the field redefinition which maps a gravitational theory formulated in the Einstein frame to the Jordan frame.

The trace of the stress tensor can be expressed as a dilatation current defined [175] as

$$\partial_\mu D^\mu =: T_\mu^\mu \quad (3.94)$$

with the definition $D^\mu \equiv T^{\mu\nu} x_\nu$ where x_ν is covariant ¹⁵.

Note that if we treated the metric as a quantum field instead of a classical background, we would obtain a new Jacobian in equation (3.90) corresponding to the field redefinition of the metric. However, this Jacobian corresponds to diagrams with closed graviton loops and scalar fields and gravitons as external lines. These diagrams are not renormalisable within quantum general relativity as they are renormalisable only for pure gravity and in the presence of matter they are non-renormalisable even at one loop (see, for example, [8, 9]). Thus one needs to address this issue in a proper quantum theory of gravity which will be discussed in Chapter 4 in the context of canonical quantisation. This is our main motivation to keep a classical background metric.

Our calculation shows that the partition function of the theory defined in the Einstein frame can be mapped to the Jordan frame in a consistent manner. When mapping the quantum field theory defined in curved space-time, one needs to take into account a Jacobian arising from the transformation of field variables. Since the transformations only involve a redefinition of dummy variables, physics cannot be affected.

¹⁵The ordinary derivative may be replaced by covariant derivatives when normal coordinates are applied because the Christoffel symbol is zero at the point chosen.

While it is obviously true that the Jacobian is related to the question of the anomaly [109–117], the presence of an anomaly is not an obstacle when changing frames as long as proper mapping is done. Furthermore, as we mentioned before, this construction is not to be confused with anomaly calculation by functional methods because we made this formula able to deal with more generic field redefinitions and the anomaly is only a special case of this formalism. If one identifies an anomaly in one frame it is just an indication that the corresponding symmetry is broken in any frame.

3.6 Boundary Terms and Conserved Currents

In this section, we integrate the previous formulation in Section 3.5 into the context of the Schwinger-Dyson equation. We will also introduce the boundary term and conserved currents interpretation.

The crucial difference at the quantum level is that *the change of measure* during the field reparameterisation $d\mu[\phi] \neq d\mu[\phi']$ is significant in both the path integral in Section 3.5. However, we should mention that the boundary terms obtained before Section 3.3 are manifested as conserved currents under the transformation or alternatively the equation of motion. This has been proven in Section 3.6.2.

In order to discuss things at the quantum level, our emphasis will turn to the Schwinger-Dyson equation which is a non-perturbative method for studying the equation of motion and conserved current within the Green's function. As we mentioned in Section 3.1, it is more appropriate to study the equivalence at the functional integral level instead of at the perturbative level. The Schwinger-Dyson equation is the quantum version of field equations of motion (EOM) and also the quantum version of Noether's current (conserved current). We start from the general Schwinger-Dyson expression; see, for example, [176–178]. Beginning with an n -point correlation function under generic field reparameterisation $\phi(x) \rightarrow \phi'(x) = \phi(x) + \epsilon(x)$ where $\epsilon(x)$ is an infinitesimal variation of the field. Suppose that we take the *measure to be the same* $d\mu[\phi] = d\mu[\phi']$ (which we will use later to highlight our result) then we obtain¹⁶

$$\begin{aligned} & \langle 0 | T \left\{ \frac{\delta}{\delta\phi(x)} \left[\int d^4x' \mathcal{L} \right] \phi(x_1) \phi(x_2) \cdots \phi(x_i) \cdots \phi(x_n) \right\} | 0 \rangle \\ &= \sum_{i=1}^n \langle 0 | T \phi(x_1) \phi(x_2) \cdots (i\delta(x - x_i)) \cdots \phi(x_n) | 0 \rangle; \end{aligned} \quad (3.95)$$

the functional derivative part is $\frac{\delta}{\delta\phi(x)} \left[\int d^4x' \mathcal{L} \right] = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right)$ which is indeed the field

¹⁶The renormalisation does not concern us here.

equation of motion and T is the time ordering. This is essentially the same as the variation method to get the field equation but within the expectation value of an n -point function. On the right hand side is the combination of the time ordered n -point function with each ‘contact point’ replaced by a delta function.

In the same way one can have the conserved current within the n -point function. If the Lagrangian is the same up to some derivative terms $\mathcal{L}(x) \rightarrow \mathcal{L}'(x) = \mathcal{L}[\phi + \epsilon \Delta \phi] = \mathcal{L}(x) + \epsilon(x) \partial_\mu \mathcal{J}^\mu(x) + (\partial_\mu \epsilon) \Delta \phi \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}$ ¹⁷ when the transformation $\phi(x) \rightarrow \phi'(x) = \phi(x) + \epsilon(x) \Delta \phi(x)$ occurs, then there is a conserved current $j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - \mathcal{J}^\mu$ where \mathcal{J}^μ is defined by $\partial_\mu \mathcal{J}^\mu \equiv \frac{\delta \mathcal{L}}{\delta \phi} \Delta \phi + (\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}) \partial_\mu \Delta \phi$ (see, for example, [178]). Thus the Schwinger-Dyson equation will be

$$\begin{aligned} & \langle 0 | T \{ \partial_\mu j^\mu(x) \phi_{a_1}(x_1) \phi_{a_2}(x_2) \cdots \phi_{a_n}(x_n) \} | 0 \rangle \\ &= \sum_{i=1}^n \langle 0 | T \phi_{a_1}(x_1) \phi_{a_2}(x_2) \cdots (-i \Delta \phi_{a_i}(x_i) \delta(x - x_i)) \cdots \phi_{a_n}(x_n) | 0 \rangle \end{aligned} \quad (3.96)$$

in which the a_i indices refer to the different types of scalar field which may be present.

Also notice that as shown in Section 3.6.2 the EOM / conserved currents are identical to the boundary terms with the exception of the infinitesimal transformation parameter. The conserved current is indeed the EOM

$$\begin{aligned} -\partial_\mu \epsilon(x) j(x)^\mu &= \epsilon(x) \partial_\mu \mathcal{J}^\mu(x) + [\partial_\mu \epsilon(x)] (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi) \\ &= \epsilon(x) [\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)})] \Delta \phi \end{aligned} \quad (3.97)$$

also the boundary terms are also shown to be identical to the conserved currents.

$$\begin{aligned} \mathcal{L}_J(x) &= \mathcal{L}_E(x) - (\partial_\mu \epsilon(x) j^\mu) \\ &= \mathcal{L}_E(x) + \partial_\mu (\text{boundary terms}). \end{aligned} \quad (3.98)$$

With this understanding, we turn to the functional integral and Schwinger-Dyson equation. We can use this framework to understand all the elements put together, including the Jacobian, boundary terms, equation of motion and conserved current. Thus we use the usual derivation of the Schwinger-Dyson equation for the field equation but now *allow the measure to be changed*. First we concern ourselves with the partition function in the formula of Section 3.5 from the Jordan frame. Of course the reverse will also be true, for the Einstein frame with the additional boundary terms in the Lagrangian and Jacobian

¹⁷The additional term $(\partial_\mu \epsilon) \Delta \phi \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}$ is due to the fact that $\epsilon(x)$ is a variable.

in the measure. The normalisation $N \rightarrow \tilde{N}$ is not of importance here,

$$Z_J = N \int d\mu[\phi] \exp \frac{i}{\hbar} \left(\int d^4x \mathcal{L}_J + \int d^4x \sqrt{-g} J_\phi \phi \right) \quad (3.99)$$

$$\begin{aligned} \rightarrow Z_E &= \tilde{N} \int d\mu[\tilde{\phi}] \exp \frac{i}{\hbar} \left(\int d^4x \mathcal{L}_E + \int d^4x \sqrt{-\tilde{g}} \tilde{J}_\phi \tilde{\phi} \right) \\ &= \tilde{N} \int \det C_{N'N} d\mu[\phi] \exp \frac{i}{\hbar} \left(\int d^4x (\mathcal{L}_J - \partial_\mu (\mathbf{boundary terms})) \right. \\ &\quad \left. + \int d^4x \sqrt{-\tilde{g}} \tilde{J}_\phi \tilde{\phi} \right) \end{aligned} \quad (3.100)$$

Following all of the previous discussions (3.97), (3.98), (3.91), (3.92) and (3.94) we transform the time ordered functional integral in the Jordan frame to the Einstein frame. We then rewrite all the Einstein frame variables in terms of the Jordan frame variables in order to make a comparison. The derivation is generally true for the case with any field redefinition between frames; here multiple fields are not included in the discussion,

$$\begin{aligned} &\langle 0 | \int d\mu[\phi] T \exp \frac{i}{\hbar} \left(\int d^4x \mathcal{L}_E + \int d^4x \sqrt{-g} J_\phi \phi \right) | 0 \rangle \\ \rightarrow &\langle 0 | \int d\mu[\tilde{\phi}] T \exp \frac{i}{\hbar} \left(\int d^4x \mathcal{L}_E + \int d^4x \sqrt{-\tilde{g}} \tilde{J}_\phi \tilde{\phi} \right) | 0 \rangle \end{aligned} \quad (3.101)$$

$$\begin{aligned} &= \langle 0 | \int d\mu[\phi] T \det C \exp \frac{i}{\hbar} \left(\int d^4x (\mathcal{L}_J - \partial_\mu \mathbf{BC term}) \right. \\ &\quad \left. + \int d^4x \sqrt{-g} J_\phi \phi \right) | 0 \rangle, \end{aligned} \quad (3.102)$$

in which we have defined $\partial_\mu \bar{D}^\mu \equiv [\int \frac{1}{2} d^4x \langle T_{\mu\nu} \rangle (\delta g^{\mu\nu})]$ according to (3.94). This will be equal to

$$\begin{aligned} &\langle 0 | \int d\mu[\phi] T \exp \left[\frac{i}{\hbar} \partial_\mu \bar{D}^\mu \right] \exp \frac{i}{\hbar} \left(\int d^4x \mathcal{L}_J + \epsilon(x) (\partial_\mu j^\mu) \right. \\ &\quad \left. + \int d^4x \sqrt{-g} J_\phi \phi \right) | 0 \rangle \end{aligned} \quad (3.103)$$

since (3.98) holds.

With the same procedure used to derive the Schwinger-Dyson current equation, we expand equation (3.103) and move the identical part to the left to cancel (3.101) and normalise using \tilde{N} . Thus the corresponding Schwinger-Dyson equation in the Einstein frame represented with Jordan frame variables for the current is

$$\begin{aligned} &\langle 0 | T \{ \partial_\mu [j^\mu + \bar{D}^\mu] \phi_{a_1}(x_1) \phi_{a_2}(x_2) \cdots \phi_{a_n}(x_n) \} | 0 \rangle \\ &= \sum_{i=1}^n \langle 0 | T \phi_{a_1}(x_1) \phi_{a_2}(x_2) \cdots (-i \Delta \phi_{a_i}(x_i) \delta(x - x_i)) \cdots \phi_{a_n}(x_n) | 0 \rangle. \end{aligned} \quad (3.104)$$

In comparison with the ordinary Schwinger-Dyson equation (3.96), we see that the field reparameterisation provides the non-conserved current. This current modifies the original conserved current of the Schwinger-Dyson equation in the Einstein frame. One novelty is

that everything is expressed in terms of total derivatives which then provides an understanding that the conserved currents, can actually in a sense be considered as boundary terms, and are changed by another current - the Jacobian, which can be regarded as further boundary terms¹⁸. Thus one may interpret the trace of the stress tensor as some boundary effects.

Using a similar procedure as above, one may also see that the quantum field equation of motion $\frac{\delta}{\delta\phi(x)}[\int d^4x' \mathcal{L}]$ in the Schwinger-Dyson formula in the Einstein frame (again one can see vice versa) is also modified to give

$$\begin{aligned} & \langle 0 | T \{ \{ \frac{\delta}{\delta\phi(x)} [\int d^4x' \mathcal{L}] + \bar{T}_\mu^\mu \} \phi(x_1) \phi(x_2) \cdots \phi(x_i) \cdots \phi(x_n) \} | 0 \rangle \\ &= \sum_{i=1}^n \langle 0 | T \phi(x_1) \phi(x_2) \cdots (i\delta(x - x_i)) \cdots \phi(x_n) | 0 \rangle \end{aligned} \quad (3.105)$$

where $\bar{T}_\mu^\mu \equiv \frac{1}{2} \int dx^4 \langle T_{\mu\nu} \rangle (\delta g^{\mu\nu})$. By comparison with the original Schwinger-Dyson equation (3.95), one can see the quantum field equation is modified by \bar{T}_μ^μ . This quantum version of the EOM, which governs the dynamics, has received an additional mass term.

Physical observables in two frames shall be identical; thus in order to make these two frames comparable, extra terms from the Jacobian effect must be taken into account when comparing one frame to the other in curved space at the quantum level. With the extra modification to the EOM or conserved currents, the quantum equation of motion (3.95) has become (3.105) and the conserved current formula (3.96) has altered to become (3.104).

As a consequence we propose a method to offset the non-conserved current by adding extra boundary terms which we will now explain.

3.6.1 Adding Extra Boundary Terms to Offset the Non-conserved Current

Physical observables or the observational results shall always be independent of the frame; the Jacobian effect arises only as a theoretical difference not as an observational difference. In order to make the theoretical comparison between the two frames correctly, we should take in to account the Jacobian related effects when rewriting and comparing the theory

¹⁸Although the trace of the stress tensor is the value in the bulk. However the relation between bulk and boundary in physics is well recognised. In addition to Ads/CFT, in classical fluid dynamics it is recognised that the surface tension (boundary) is the trace of the stress tensor in the bulk. Also in the cancellation of the gauge and gravitational anomalies in black holes, these anomalies can be regarded as Hawking radiation (blackbody radiation) which is a boundary effect [179, 180].

to the other frame. An alternative viewpoint to reconcile this situation is the following: In terms of the conserved current being changed by the non-conserved current (3.104), one may observe that the origin of the the current is the extra total derivatives as we formulated previously as boundary terms. Thus one may consider that the additional effect originates from some boundary effects and therefore in order to cure this one may add some counter boundary terms which will offset these effects.

In analogy to [131, 132], where it is proposed to remove the anomaly by adding specific higher order terms apart from the normal Einstein-Hilbert action in L_{eff} , we can also add specific terms. In our case some extra boundary terms can offset the Jacobian related effect, which includes the anomaly as a special case as mentioned before in Section 3.5, and the anomaly discussion in terms of Schwinger-Dyson equation can also be found [181–183]. The discrepancy between the classical and quantum field equation produces the anomalous Jacobian. This gives rise to the origin of the anomaly. They define the quantum field equation by utilising the property of the functional integral in which the integration of functional derivatives will vanish. Yet the classical field equation comes from demanding that the variation of the Lagrangian is zero. Then the difference between these two will be the Jacobian effect rendering extra terms in the field equation at the quantum level.

This subsection proposes a scheme to offset the Jacobian related effect by adding some additional boundary terms because as we showed in equation (3.104) the Jacobian related effect is actually a total derivative term, thus a boundary term, which modifies the original conserved current which is also a boundary term as shown in Section 3.6.2. Since they can all be interpreted as currents or boundary terms we propose that by adding some counter boundary terms, it is viable to cancel the Jacobian related effect and this will not influence the original EOM in the bulk. Though the purpose is different, this is analogous to the Gibbons-Hawking term [133] in which a boundary term is added to obtain the correct field equation when in the open manifold.

The terms added into the boundary are basically those in the dilatation current with opposite sign (3.94) before taking the vacuum expectation value and renormalisation. Recall the definition $\partial_\mu D^\mu =: T^\mu_\mu$ with the definition $D^\mu \equiv T^{\mu\nu} x_\nu$ where x_ν is the ordinary covariant coordinate vector. The extra boundary term will be

$$\begin{aligned} \mathcal{B}^\mu \equiv & -x_\nu \{ \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{2} g^{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi \\ & - \xi (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) \phi^2 + \xi [g^{\mu\nu} \square \phi^2 + \nabla^\mu \nabla^\nu \phi^2] \}. \end{aligned} \quad (3.106)$$

Therefore in equation (3.104) we will have an additional current (boundary term) $\partial_\mu \mathcal{B}^\mu$

to offset the Jacobian effect - the dilatation current. Hence the modification will have no effect on the original conserved current or the EOM at the quantum level. In brief

$$\partial_\mu[j^\mu + \bar{D}^\mu + \bar{\mathcal{B}}^\mu] = 0 \quad (3.107)$$

Here one can see how the effect of boundary terms may play a role in cancelling the Jacobian related effects.¹⁹ Importantly this also shows the nature of these Jacobian related effects as boundary terms, which cannot be discarded away naively in curved space.

In the next subsection we elaborate on what we mean by the boundary term, equation of motion and conserved current can be presented identically and we will return back to the main line of discussion later.

3.6.2 Supplementary Derivation: Relation of EOM, Conserved Current and Boundary Terms

In this section we will show the relation between the field equation (EOM), the conserved current and the boundary terms. The conserved currents are identical to the boundary terms. The boundary terms are produced when the Lagrangian undergoes certain transformations or field redefinitions. The classical action is then identical up to surface terms and so is the Lagrangian up to boundary terms after the transformation.

Beginning with some familiar results from Noether's theorem, the EOM is the same and the action is identical up to a surface term. In the case of the field redefinition, the mathematical expression of the action may change under the new set of variables and these two actions are indeed formally identical at the classical level. The Lagrangian is also identical up to some total derivative terms

$$\mathcal{L}(x) \rightarrow \mathcal{L}'(x) = \mathcal{L}[\phi + \epsilon\Delta\phi] = \mathcal{L}(x) + \epsilon(x)\partial_\mu\mathcal{J}^\mu(x) + (\partial_\mu\epsilon)\Delta\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \quad (3.108)$$

The additional term $(\partial_\mu\epsilon)\Delta\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}$ is due to $\epsilon(x)$ being a local parameter (if we only considered a constant parameter this term would not exist). To be more precise, under the field transformation $\phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$, the last two terms in equation (3.108), with the definition $\partial_\mu\mathcal{J}^\mu \equiv \frac{\delta\mathcal{L}}{\delta\phi}\Delta\phi + (\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)})\partial_\mu\Delta\phi$, can be rewritten as $\Delta\{\epsilon(x)\mathcal{L}\}$

¹⁹In some other contexts, the significance of boundary terms in calculating the stress tensor in the finite temperature system is in [184]

where,

$$\Delta\{\epsilon(x)\mathcal{L}\} = \epsilon(x)\partial_\mu\mathcal{J}^\mu(x) + (\partial_\mu\epsilon)\Delta\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \quad (3.109)$$

$$\begin{aligned} &= \frac{\delta\mathcal{L}}{\delta\phi}(\epsilon(x)\Delta\phi) + (\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)})\partial_\mu(\epsilon(x)\Delta\phi) \\ &= \frac{\delta\mathcal{L}}{\delta\phi}(\epsilon(x)\Delta\phi) + \partial_\mu(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\epsilon(x)\Delta\phi) - \epsilon(x)\Delta\phi\partial_\mu(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}) \\ &\quad + (\partial_\mu\epsilon)\Delta\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} - (\partial_\mu\epsilon)\Delta\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \\ &= \epsilon(x)\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi) + \epsilon(x)[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})]\Delta\phi \\ &\quad + (\partial_\mu\epsilon)\Delta\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}; \end{aligned} \quad (3.110)$$

in the third line we have employed a trick by adding and removing the term $(\partial_\mu\epsilon)\Delta\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}$ simultaneously.

The $\epsilon(x)[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})]\Delta\phi$ term is indeed the Euler-Lagrange field equation of motion (EOM). Keeping this term unchanged (see (3.116) below), we move the rest of the terms to the first line in (3.109); then $(\partial_\mu\epsilon)\Delta\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}$ is cancelled, and we assign the rest of them as $-\partial_\mu\epsilon(x)j(x)^\mu$; see (3.117) below, corresponding to a conserved current for a specific field transformation,

$$-\partial_\mu\epsilon(x)j(x)^\mu \equiv \epsilon(x)\partial_\mu\mathcal{J}^\mu(x) - \epsilon(x)\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi) \quad (3.111)$$

$$= \epsilon(x)[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})]\Delta\phi \quad (3.112)$$

The second line (3.112) is the EOM.

In conclusion, we collect all the formulae:

$$\mathcal{L}(x) \rightarrow \mathcal{L}'(x) = \mathcal{L}[\phi + \epsilon\Delta\phi] \quad (3.113)$$

$$= \mathcal{L}(x) + \epsilon(x)\partial_\mu\mathcal{J}^\mu(x) + (\partial_\mu\epsilon)\Delta\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \quad (3.114)$$

$$= \mathcal{L}(x) + \Delta\{\epsilon(x)\mathcal{L}\} \quad (3.115)$$

$$= \mathcal{L}(x) + \epsilon(x)[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})]\Delta\phi \quad (3.116)$$

$$= \mathcal{L}(x) - \partial_\mu\epsilon(x)j(x)^\mu. \quad (3.117)$$

There are two observations following this identity. First, as we expected, the conserved current is actually identical to the EOM; they are just different expressions obtained from the different variation parameters. Importantly the second observation is that the terms $\epsilon(x)\partial_\mu\mathcal{J}^\mu(x) - \epsilon(x)\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi)$ as a conserved current are indeed the ‘boundary terms’ in equation (3.21)

$$\mathcal{L}_J = \mathcal{L}_E + \partial(\text{boundary terms})$$

so that $-\partial_\mu \epsilon(x) j(x)^\mu$ is exactly our boundary term $\partial_\mu(\text{boundary terms})$. Accordingly, *the EOM, conserved currents and boundary terms can all be identified with each other.*

This section provides the proof of equation (3.97) and (3.98).

3.7 Conclusion

We have shown how to map gravitational theories formulated in the Jordan frame to the Einstein frame at the quantum field theory level. While it has been known that the theories were equivalent up to a boundary term at the classical level, we have shown that there is a new term produced from the Jacobian at the quantum level which is properly discussed using quantum fields in curved space-time. The non-trivial Jacobian must be taken into account when making a comparison of physical observables between frames. In addition, we also found that the Jacobian related quantities, the conserved current and the field equation can all be represented as boundary terms. Accordingly, the Jacobian related effect can be offset by adding some extra boundary terms. The significance of the Jacobian is shown in both the path integral and canonical quantisation approaches. There is no such issue when remaining in one frame and no comparison arises. In addition, the Jacobian originates from the measure in the path integral, hence it is a purely quantum effect.

The effect of the Jacobian not only manifests itself in the path integral but also importantly in the canonical quantisation approach. The Hamiltonian constraints are of central importance in canonical quantisation, within which the variables convert into operators in either the position representation or the momentum representation. The Hamiltonian constraints originate from reparameterisation invariance according to [14] and they also generate redundancy - normally referred to as a symmetry. Importantly there is a Jacobian factor produced in the Hamiltonian constraint when applying field reparametrisation of the Hamiltonian in terms of the variables in the other frame. The Jacobian must be taken into account when comparing between frames; this holds under quantisation because all the generalised coordinate and momentum have become operators. Therefore the Jacobian factor is important in both path integral and canonical quantisation method.

In a quantum field theory, fields are dummy variables and summed over; in so far as there are different measures in the respective frames which need to be dealt with. This is particularly obvious when using the path integral quantisation formulation. Field redefinitions cannot affect the calculation of observables. The physical equivalence of the frames is obvious if the field redefinitions are done properly. Our results can easily be extended to

any gravitational theory. In conclusion, we reaffirm that the frame transformation cannot affect the calculations of observables as long as the proper boundary terms and Jacobian terms are taken into account.

Chapter 4

Hořava's Gravity meets Canonical Quantisation

4.1 Introduction

In this chapter, we shall consider an exciting recent approach to quantum gravity developed by Soo and Yu [1, 185]. This approach consists in a fusion between Hořava's gravity [10] which has had a considerable impact in the last few years with the well established framework of canonical quantisation of gravity. Soo and Yu's model is exciting because it could potentially address the well known issues of quantum gravity models: the lack of renormalisability, issues with unitarity and the problem of time. We first review the canonical quantisation procedure for General Relativity, then introduce Hořava's gravity. We then review the canonical quantisation procedure applied to Hořava's gravity. Then we proceed to a symmetry reduction and study a system with spherical symmetry. Furthermore, we derive the quantum Schrödinger equation and find analytic solutions of wave function.

4.2 Review of General Formalisms of Canonical Quantisation and Geometrodynamics

4.2.1 Classical Formalism

First we review the well-known conventional canonical quantisation scheme developed in [30, 31]. To obtain a quantum version of general relativity¹, we start from the Einstein-

¹We change the convention from $(+, -, -, -)$ to $(-, +, +, +)$ in this chapter because in the ADM decomposition scheme there are three spatial dimensions and it is more convenient.

Hilbert action

$$\mathcal{S}_{\text{gravity}} = \frac{c^3}{16\pi G} \int d^4x (R - 2\Lambda) \sqrt{-g}. \quad (4.1)$$

That is also called *Geometrodynamics* which is known to be equivalent to *gauge dynamics* if the canonical variables are the connection and the vielbein [49–55].

We assume that the manifold is equipped with a codimension one foliation structure. Then the Arnowitt-Deser-Misner (ADM) decomposition of the metric is used to *foliate* the manifold into $3 + 1$ spacetime where q_{ij} is the three dimensional metric. Note that $N(x)$ (which is called the lapse) and $N^i(x)$ (which is called the shift) are both Lagrangian multipliers in front of the respective constraint. The line element becomes

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 (dx^0)^2 + q_{ij} (dx^i + N^i dx^0) (dx^j + N^j dx^0), \quad (4.2)$$

where the metric is given by,

$$g_{\mu\nu} = \begin{bmatrix} q_{ij} N^i N^j - N^2 & q_{ij} N^i \\ q_{ij} N^j & q_{ij} \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} \frac{-1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^j}{N^2} & q^{uj} - \frac{N^i N^j}{N^2} \end{bmatrix}. \quad (4.3)$$

We substitute this decomposition into the Einstein-Hilbert action (4.1),

$$S = \int dx^0 d^3x (\tilde{\pi}^{ij} (\partial_0 q_{ij}) - N_i H^i - N H) + \text{boundary term}, \quad (4.4)$$

where we ignore the boundary term. However, it is straightforward to add the boundary term if necessary. The conjugate momentum is given by

$$\tilde{\pi}^{ij} = \frac{\sqrt{q} c^3}{16\pi G} [q^{ij} K_l^l - K^{ij}]$$

which is a tensor density of weight 1, and K_{ij} is the extrinsic curvature of the three dimension space defined by $K_{ij} = \frac{1}{2N} (-\partial_0 q_{ij} + \nabla_i N_j + \nabla_j N_i)$.² Because of the ADM decomposition, the spacetime manifold is now $3 + 1$ and all the relevant variables (such as $R_{jkl}^i, \Gamma_{ij}^k, K_{ij}$) are three-dimensional³. General covariance is preserved since the codimension one can be chosen covariantly. However, in the sequel we will explore the Soo and Yu's new model [1, 185] which preserves general covariance for the three dimensions only.

Next follows the canonical quantisation of the action obtained above. One quantises the theory in the superspace which is comprised of all equivalence classes of three metrics q_{ij} for each coordinate point of space [30, 31]. The super-momentum H^i and super-Hamiltonian H both vanish as constraints. These are first class constraints according

²Tensors with a density weight +1 will be denoted with a tilde from now on.

³Lower case Latin indices represent spatial indices and upper case letters are four-dimensional spacetime indices.

to [186–190]. A first class constraint leads to redundancy (‘gauge’) transformations and one can thus see a gauge theory as a subclass of a constrained system. In other words, the constraint leads to the notation of symmetry. The super-momentum constraint is related to the four dimensional diffeomorphism invariance while the super-Hamiltonian constraint is related to the invariance under time evolution and thus energy conservation. The explicit form of these two constraints are

$$H_i = -\frac{c^3}{8\pi G} \nabla_j \tilde{\pi}_i^j \approx 0 \quad (4.5)$$

$$H = \frac{8\pi G}{c^3 \sqrt{q}} (q_{ik} q_{jl} + q_{il} q_{jk} - q_{ij} q_{kl}) \tilde{\pi}^{ij} \tilde{\pi}^{kl} - \frac{c^3 \sqrt{q}}{16\pi G} (R^{(3)} - 2\Lambda) \quad (4.6)$$

$$= \frac{c^3 \sqrt{q}}{16\pi G} [K^{ij} K_{ji} - (K^l_l)^2 - R^{(3)} + 2\Lambda] \quad (4.7)$$

$$= \frac{c^3 \sqrt{q}}{16\pi G} [Tr K^2 - (Tr K)^2 - R^{(3)} + 2\Lambda] \approx 0, \quad (4.8)$$

where $R^{(3)}$ is three dimensional Ricci scalar. Again, we point out in 3.4.3 that the zero point energy cannot be set arbitrarily since it changes the constant part of super-Hamiltonian constraint Λ , the cosmological constant. As we will see later, the super-Hamiltonian constraint plays a double role in both generating the symmetry and governing the dynamics. That causes some conceptual problems, since the Hamiltonian is meant to generate the time evolution but it is also the constraint and must be zero. Thus the time evolution is always frozen.

According to [14], the Poisson bracket algebra of these two constraints are given by [186]:

$$\{H[N], H[N']\} = H[-\sigma q^{ij} (N \partial_j N' - N' \partial_j N)] \quad (4.9)$$

$$\{H[N^i], H[N']\} = H[N^i \partial_i N] \equiv H[\mathcal{L}_{N^i} N] \quad (4.10)$$

$$\{H[N^i], H[N'^j]\} = H[N^i N'^j - N'^j N^i] \equiv H[\mathcal{L}_{N^i} N'^j], \quad (4.11)$$

where the smearing functions are defined by $H[N] \equiv \int d^3x N(x) H(x)$, and $H[N^i] \equiv \int d^3x N^i(x) H_i(x)$, and where σ is the signature of the embedded spacetime⁴. This algebra is not a Lie algebra since (4.9) contains a structure function q^{ij} instead of a structure constant. However, the (4.11) is indeed a Lie Algebra.

⁴Note that one is not summing over the indices in the notation of [14]. For instance, if the argument in the square bracket has no free index, it represents Hamiltonian constraint. On the other hand the one with a free index represents a momentum constraint.

4.2.2 Degrees of freedom

It is essential to clarify the degrees of freedom for the gravitational field. The gravitational field is subject to two types of constraints in order to determine the exact number of dynamical variables. Because of the symmetry properties of q_{ij} and $\tilde{\pi}^{ij}$, there are six degrees of freedom for the $(q_{ij}, \tilde{\pi}^{ij})$ pair. Nevertheless, they are constrained by the super-momentum and super-Hamiltonian which fix three and one degrees of freedom, respectively. Therefore, the residual field has two degrees of freedom.⁵

4.2.3 The Intrinsic and Extrinsic Curvature

We introduce the extrinsic curvature or second fundamental form in this section. The intrinsic curvature R^i_{jkl} of three dimensional space can be obtained from the Christoffel symbol (or affine connection) which is deduced from the metric,

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}) \quad (4.12)$$

$$R^p_{qij} = \partial_i \Gamma^p_{jq} - \partial_j \Gamma^p_{iq} + \Gamma^p_{im} \Gamma^m_{jq} - \Gamma^p_{jm} \Gamma^m_{iq}, \quad (4.13)$$

while the extrinsic curvature K_{ij} is defined by the variation of the vector orthogonal to three surfaces under parallel transport, $x^i \rightarrow x^i + \delta x^i$,

$$\delta n_i = -K_{ij} dx^j$$

$$K_{ij} = \frac{1}{2N}(-\partial_0 q_{ij} + \nabla_i N_j + \nabla_j N_i) \quad (4.14)$$

$$= \frac{1}{2N}(\frac{dq_{ij}}{dt} - \mathcal{L}_{\vec{N}} q_{ij}). \quad (4.15)$$

4.2.4 Canonical Quantisation of General Relativity and Wave Function

The canonical quantisation of gravity does not require any background spacetime, and it leads to a background-independent quantum theory. We rewrite the action (4.1) with constraints as

$$S = \int dx^0 d^3x (\tilde{\pi}^{ij} \partial_0 q_{ij} - N_i H^i - N H). \quad (4.16)$$

The Poisson bracket of the conjugate pair takes the form

$$\begin{aligned} \{q_{ij}(\vec{x}), \tilde{\pi}^{kl}(\vec{y})\} &= \frac{1}{2}(\delta_i^k \delta_j^l + \delta_j^k \delta_i^l) \delta^3(\vec{x} - \vec{y}) \\ &= [\delta_i^k \delta_j^l] \delta^3(\vec{x} - \vec{y}). \end{aligned} \quad (4.17)$$

⁵In general, the number of degrees of freedom of a field in N spacetime dimensions is: $\frac{N(N-1)}{2} - N = \frac{N(N-3)}{2}$. Therefore, the lowest dimensionality with a non-trivial degree of freedom is 4.

We now quantise the fields by promoting them to operators and by replacing the Poisson bracket by a commutator as in the conventional canonical quantisation procedure⁶:

$$\{ , \}_{P.B.} \longrightarrow \frac{1}{i\hbar} [,] \quad (4.18)$$

$$q_{ij} \longrightarrow \hat{q}_{ij} \quad (4.19)$$

$$\tilde{\pi}^{ij} \longrightarrow \hat{\pi}^{ij}. \quad (4.20)$$

In the metric representation $\Psi[q_{ij}]$ is the quantum wave function and it is a functional of q_{ij} . The quantum wave function can be understood as the wave function obeying the quantum Hamiltonian, which will be discussed later on in this section. On the other hand, $\Psi[q_{ij}]$ can also be expressed as a path integral and thus the amplitude composed by the sum over all four dimension geometries weighted by $\exp(iS)$ with the space-like metrics as their boundaries. This is in line with the definition of wave functions which specify the universe or a quantum gravity system [191]. The wave function is defined as

$${}_{\text{initial}}\langle q_{ij} | q_{ij} \rangle_{\text{final}} \equiv \Psi[q_{ij}] \equiv n \int \delta g(x) \exp(iS[g]) \quad (4.21)$$

where n is a the normalisation and S is the action. It is also the correlation between observables to be expected within these state.

The structure at each time is probed in three dimensions rather than in the whole geometry. This is in analogy to the particle wave function in quantum mechanics. The whole spacetime is the result of path integral. This is one of the reasons why the arena of quantum gravity is three dimensional.

The canonical pair must also be replaced by operators either in the q_{ij} or the $\tilde{\pi}^{ij}$ representation. We will mainly work using the q_{ij} representation,

$$\hat{q}_{ij}\Psi[q_{ij}] = q_{ij}\Psi[q_{ij}], \quad \hat{\pi}^{ij}\Psi[q_{ij}] = \frac{\hbar}{i} \frac{\delta}{\delta q_{ij}} \Psi[q_{ij}]. \quad (4.22)$$

The constraints are therefore replaced in terms of the operator form respectively,

$$H_i[\hat{q}_{ij}, \hat{\pi}^{ij}]\Psi[q_{ij}] \rightarrow \hat{H}_i[\hat{q}_{ij}, \frac{\hbar}{i} \frac{\delta}{\delta q_{ij}}]\Psi[q_{ij}] = 0. \quad (4.23)$$

$$H[\hat{q}_{ij}, \hat{\pi}^{ij}]\Psi[q_{ij}] \rightarrow \hat{H}[\hat{q}_{ij}, \frac{\hbar}{i} \frac{\delta}{\delta q_{ij}}]\Psi[q_{ij}] = 0. \quad (4.24)$$

Accordingly, the wave function has a double interpretation: the solution for the Hamiltonian and the amplitude of the path integral.

⁶From now on, we will omit the hat notation if no ambiguity occurs.

4.2.5 Wheeler-DeWitt Equation and Super-momentum Constraint

The explicit form of the super-momentum constraint is

$$H_i \Psi[q_{ij}] = -\frac{c^3}{8\pi G} \nabla_j \frac{\delta \Psi}{\delta q_{ij}} = 0. \quad (4.25)$$

It generates general coordinate transformations, thus we can have

$$\Psi[q'_{ij}] = \Psi[q_{ij}]$$

where q_{ij} is related to q'_{ij} by a three dimensional diffeomorphism (general coordinate transformation). Thus, $\Psi[q_{ij}]$ is a functional in the superspace and its arguments are all equivalence classes of the three dimensional metric.

The super-Hamiltonian constraint is

$$H = \frac{8\pi G}{c^3 \sqrt{q}} (q_{ik} q_{jl} + q_{il} q_{jk} - q_{ij} q_{kl}) \tilde{\pi}^{ij} \tilde{\pi}^{kl} - \frac{c^3 \sqrt{q}}{16\pi G} (R^{(3)} - 2\Lambda) = 0, \quad (4.26)$$

whereas the quantised Super-Hamiltonian constraint is

$$\left\{ \frac{8\pi G}{c^3 \sqrt{q}} (q_{ik} q_{jl} + q_{il} q_{jk} - q_{ij} q_{kl}) \hbar^2 \frac{\delta}{\delta q_{ij}} \frac{\delta}{\delta q_{kl}} - \frac{c^3 \sqrt{q}}{16\pi G} (R^{(3)} - 2\Lambda) \right\} \Psi[q_{ij}] = 0 \quad (4.27)$$

or alternatively,

$$\left\{ \frac{16\pi G}{c^3 \sqrt{q}} G_{ijkl} \hbar^2 \frac{\delta}{\delta q_{ij}} \frac{\delta}{\delta q_{kl}} - \frac{c^3 \sqrt{q}}{16\pi G} (R^{(3)} - 2\Lambda) \right\} \Psi[q_{ij}] = 0. \quad (4.28)$$

This is the well-known Wheeler-DeWitt equation where $G_{ijkl} = \frac{1}{2}(q_{ik} q_{jl} + q_{il} q_{jk} - q_{ij} q_{kl})$ is the supermetric. Symbolically, this equation can be written as

$$\left[\frac{\delta^2}{\delta \mathcal{G}^2} + (R^{(3)} - 2\Lambda) \right] \Psi[q_{ij}] = 0, \quad (4.29)$$

where $\frac{\delta^2}{\delta \mathcal{G}^2}$ represents the kinematic term $\frac{16\pi G}{c^3 \sqrt{q}} G_{ijkl} \hbar^2 \frac{\delta}{\delta q_{ij}} \frac{\delta}{\delta q_{kl}}$. This is the equation that governs the quantum behaviour of gravity. Since $H\Psi[q_{ij}] = 0$, the ‘time evolution’ is given by

$$\exp(-\frac{i}{\hbar} x^0 \int H d^3x) \Psi[q_{ij}] = \Psi[q_{ij}], \quad (4.30)$$

which leads to the states being frozen rather than evolving. At first sight, it looks like there is no time in that framework of quantum gravity. However, one can find an intrinsic time in this framework [30]. One can rewrite the supermetric as $G^{ijkl} \delta g_{ij} \delta g_{kl} = -(\delta \xi)^2 + \frac{3}{32} \xi^2 \bar{G}_{AB} \delta \xi^A \delta \xi^B$, i.e. $G^{\{ij\}\{kl\}} = \text{diag}(-1, \frac{3}{32} \xi^2 \bar{G}_{AB})$, ($A, B = 1, 2, 3, 4, 5$). \bar{G}_{AB} is positive definite and the supermetric has the signature $(-, +, +, +, +, +)$. The minus sign in general refers to the intrinsic time of Wheeler-DeWitt equation ($\xi = \sqrt{32/3} (\det q)^{\frac{1}{4}}$)

⁷, and $\delta\xi^A \perp \delta\xi$. As a result, the super-Hamiltonian has ξ which fulfils the role of the direction of time,

$$\left[-\frac{\delta^2}{\delta\xi^2} + \frac{32}{3\xi^2}\bar{G}^{AB}\frac{\delta}{\delta\xi^A}\frac{\delta}{\delta\xi^B} + \frac{3\xi^2}{32}(R^{(3)} - 2\Lambda)\right]\Psi[q_{ij}] = 0. \quad (4.31)$$

As we will discuss later, we can assign certain degrees of freedom to the intrinsic time in the quantum regime. A physical requirement is that the time must depict dynamics and define the probability properly. Furthermore, its connection to the classical passage of time is another challenge since any physical quantity in the quantum regime must have some form of a classical limit. Otherwise its definition would be purely mathematical, which is the case in (4.31).

4.3 New Approach to the Canonical Quantisation of Hořava's Gravity

In this section we will review a new model for quantum gravity recently proposed by Soo and Yu [1, 185]. This approach is interesting as, according to its author, it could solve the issues of quantum gravity discussed above- renormalisability, unitarity preservation and problem of time. It merges canonical quantisation with the model of Hořava for gravity [10] with master constraints to cure the inconsistent constraint of Hořava's Gravity. In addition they use new set of variables; that is able to separate a degree of freedom which will later on be identified as intrinsic time in quantum regime. This feature allows the framework to equip a intrinsic time explicitly, that is important in defining probability and dynamics. Also the consequent Wheeler-DeWitt equation has become Schrödinger type with first order in time derivative only, that guarantees positive probability. Inherited from Hořava gravity, that insures the the theory is renormalisable from the power counting point of view. In addition unitarity is preserved because this framework possesses only three dimensional diffeomorphism invariance and thus adds only higher order spatial derivatives to achieve renormalisability, thus the unitarity is not compromised. We first review Hořava's model and then consider Soo and Yu's quantisation procedure.

4.3.1 Hořava's gravity

The new feature of Hořava's gravity [10] is the anisotropic scaling of spacetime: spatial coordinates scale as $x \rightarrow bx$ while the time one scales as $t \rightarrow b^z t$ where b is anisotropically

⁷ ξ should not be confused with the Killing vectors.

scaled with the dynamical critical exponent z . Thus this theory only has spatial diffeomorphism invariance. In order to fulfil power counting renormalisability, one can show that in $3 + 1$ dimensions $z = 3$ must be chosen. Hořava contends that the model has a UV fix point, in analogy to what happens in asymptotically safety gravity [34–40]. In this construction, the canonical dimensions of the physical variables are different from the usual ones. In Hořava’s case, the dimensionality is shown:

$$[N] = 0 \quad (4.32)$$

$$[N_i] = z - 1 \quad (4.33)$$

$$[q_{ij}] = 0 \quad (4.34)$$

$$[dt d^3x] = -3 - z \quad (4.35)$$

$$[\partial_t] = z \quad (4.36)$$

$$[\kappa] = \frac{z - 3}{2} \quad (4.37)$$

$$[c] = 2, \quad (4.38)$$

where N is lapse and N_i is shift. As an intriguing result is that the coupling constant $\kappa = \frac{8\pi G}{c^3}$, which is the Newtonian constant, is dimensionless in a $3 + 1$ spacetime when $z = 3$ [10]. The kinetic part of the action (4.4) from (4.7) is

$$S_{\text{Kinetic}} = \frac{2}{\kappa'^2} \int dt d^3x \sqrt{g} N (K_{ij} K^{ij} - \lambda (K^l_l)^2), \quad (4.39)$$

with $K_{ij} = \frac{1}{2N} (-\partial_0 q_{ij} + \nabla_i N_j + \nabla_j N_i)$ and λ is the parameter subject to quantum effect to depict deformation from Einstein gravity. Thus the corresponding supermetric is $G_{ijkl} = \frac{1}{2} (q_{ik} q_{jl} + q_{il} q_{jk}) - \frac{\lambda}{3\lambda - 1} q_{ij} q_{kl}$ when $\lambda = \frac{1}{3}$ it is the ordinary supermetric corresponding to Einstein gravity. The action is invariant under three dimensional diffeomorphism. Note that $\kappa' = \sqrt{32G\pi c}$, which is not to be confused with $\kappa = \frac{8\pi G}{c^3}$ which will be used later on.

One observes that because of the dimensionality of $dt d^3x$ which is $-3 - z = -6$, the action is power counting renormalisable. Furthermore, the potential term is allowed to contain dimension 6 operators. There is a large number of independent operators allowed, the proliferation of coupling constants will make calculations difficult. For pragmatic reasons, we limit the number of operators, as proposed by Hořava, to those used for non-equilibrium critical phenomena and quantum critical systems. The potential will thus be in a specific form called *detailed balance*. With G_{ijkl} as supermetric. Therefore, the Hamiltonian density is proportional to

$$\frac{G_{ijkl}}{\sqrt{q}} (\pi^{ij} \pi^{kl} + \frac{\delta W}{\delta q_{ij}} \frac{\delta W}{\delta q_{kl}}), \quad (4.40)$$

with the first term being kinetic term and the second term is the potential term, where

$$W \equiv \int d^3x [\sqrt{q}(a R^{(3)} - \Lambda) + g\tilde{\epsilon}^{ikl}\Gamma_{im}^l\partial_j\Gamma_{kl}^m + \frac{2}{3}\Gamma_{im}^l\Gamma_{jn}^m\Gamma_{kl}^n]. \quad (4.41)$$

It takes the form of a three dimensional Einstein-Hilbert action with cosmological constant in the regime of lower curvature according to [10]. The lower case Latin indices represent spatial indices. The cosmological constant is Λ , a and g are dimensionless coupling constants, and the last two terms form the three dimensional Chern-Simons action.

The term $\frac{G_{ijkl}}{\sqrt{q}}\frac{\delta W}{\delta q_{ij}}\frac{\delta W}{\delta q_{kl}}$ is the potential satisfying the detailed balance condition with W as defined above in (4.41). Also $\frac{\delta W}{\delta q_{ij}}$ is a third order spatial derivative and according to [10] it is uniquely chosen to be the Cotton-York tensor,

$$C^{ij} = \epsilon^{ikl}\nabla_k(R_l^j - \frac{1}{4}R\delta_l^j), \quad (4.42)$$

which is the most general symmetric second rank tensor containing third order spatial metric derivatives, as requested by the detail balance condition.

The general form of Hořava's gravity action can be understood as follows,

$$S = \int dt d^3x \sqrt{g} \left[\sum_{[\lambda_J]=6} \lambda_J \mathcal{O}^J + \sqrt{g} \sum_{[\lambda_A]<6} \lambda_A \mathcal{O}^A \right]. \quad (4.43)$$

The first part is the operators with dimension equal to 6 and it satisfies the detail balance condition. It becomes important at high energies. The latter part describes the deformation of the operator and becomes important at low energies. In addition, the coupling constants λ_A and λ_J correspond to a , Λ and g in (4.41) are subject to the renormalisation group flow. Note that the Einstein operators are of dimension less than 6 in (4.43) while the Chern-Simons terms are dimension 6 operators in (4.43), which leads to power counting renormalisability.

4.3.2 Canonical quantisation approach to Hořava's gravity

We start from the conformal decomposition explored by York [192, 193] of the spatial three dimensional metric $q_{ij} = \phi\bar{q}_{ij}$ with and with the choice of $\phi = q^{\frac{1}{3}}$ from [1], which will be proved to allow separation of temporal degree of freedom later. It leads to

$$\bar{q}_{ij} \equiv q^{-\frac{1}{3}}q_{ij} \quad (4.44)$$

where $q \equiv \det[q_{ij}]$. It is one of the generalised coordinates. Its conjugate momentum is given by

$$\bar{\pi}^{ij} \equiv q^{\frac{1}{3}}[\tilde{\pi}^{ij} - \frac{q^{ij}}{3}\pi]. \quad (4.45)$$

The other canonical pair is

$$\ln q^{\frac{1}{3}} \quad (4.46)$$

and its conjugate momentum is the trace of $\tilde{\pi}^{ij}$,

$$\pi \equiv q_{ij} \tilde{\pi}^{ij} \quad (4.47)$$

This choice leads to a clean separation of mutually commuting canonical pairs, with the symplectic potential given by

$$\int d^3x \tilde{\pi}^{ij} \delta q_{ij} = \int d^3x (\tilde{\pi}^{ij} \delta \bar{q}_{ij} + \pi \delta \ln q^{\frac{1}{3}}). \quad (4.48)$$

The Poisson brackets are given by

$$\{\bar{q}_{kl}(x), \tilde{\pi}^{ij}(x')\} = P_{kl}^{ij} \delta(x, x') \quad (4.49)$$

$$\{\ln q^{\frac{1}{3}}(x), \pi(x')\} = \delta(x, x'), \quad (4.50)$$

where $P := \frac{1}{2}(\delta_k^i \delta_l^j + \delta_l^i \delta_k^j) - \frac{1}{3} \bar{q}^{ij} \bar{q}_{kl}$ is a trace free projector. We will use these variables to quantise the theory. Also we shall see that $\ln q^{\frac{1}{3}}$ which is a degree of freedom that decouples from other degrees of freedom, can be identified as time [1].

The action is given by

$$\int [\tilde{\pi}^{ij} \partial_t q_{ij} - N^i H_i - m M] d^3x dt, \quad (4.51)$$

where

$$M \equiv \int d^3x (H^2 / \sqrt{q}) = 0, \quad (4.52)$$

and it is master constraint. The Hamiltonian constraint is replaced by the master constraint M . m is the Lagrangian multiplier. Because M does not generate a symmetry according to [56, 194], m is not involved into the dynamics. This master constraint [56, 194] does not need to be interpreted as the generator of unphysical time-development. It only determines the dynamics. Therefore, there is no ‘multi-fingered time evolution’. In addition, the time evolution is not frozen as we will see later.

This master constraint also solves the problems of inconsistent constraints in the non-projectable version of Hořava’s gravity [56]. In addition, this constraint results in a first class constraint instead of second class constraint which cures another issue of the original Hořava proposal namely the degeneracy of the metric [10, 195–199].

The new first class algebra obtained from the master constraint is

$$\{M, M\} = 0 \quad (4.53)$$

$$\{H_i[N^i], M\} = 0 \quad (4.54)$$

$$\{H_i[N^i], H_j[N'^j]\} = H_i[\mathcal{L}_{N^j} N'^i], \quad (4.55)$$

which processes only a three dimensional diffeomorphism, both on and off shell. Notice that M decouples from H_i and, unlike conventional geometrodynamics in 4.2.1, has no problem with ‘structure functions’. It is indeed a Lie algebra with a structure constant. To be precise, the total constraint of geometrodynamics $NH + H_k[N^k]$ is replaced by $m(t)M + H_k[N^k]$, which generates only three dimensional diffeomorphism invariance as a fundamental symmetry in this new model. The four dimensional diffeomorphism is recovered as an emergent symmetry in the semiclassical regime [1] and classical general relativity is recovered in that regime as well. The dynamics are dictated by H which is encoded in M and generates no further symmetry. For an arbitrary functional $f(q_{ij}, \tilde{\pi}^{ij})$ it is important to understand what kind of symmetry has been generated through the constraint by studying their Poisson bracket,

$$\{f(q_{ij}, \tilde{\pi}^{ij}), m(t)M + H_k[N^k]\} |_{M=0 \Leftrightarrow H=0} \approx \{f, H_k[N^k]\} = \mathcal{L}_{N^i} f. \quad (4.56)$$

Unlike previously, in geometrodynamics only N is generating the symmetry, while $m(t)$ plays no role in the dynamics and therefore there is no multi-fingered time.

The above-mentioned ultra-local DeWitt supermetric G_{ijkl} introduced in Section 4.2 is now in the form of

$$G_{ijkl} = \frac{1}{2}(q_{ik}q_{jl} + q_{il}q_{jk}) - \frac{\lambda}{3\lambda - 1}q_{ij}q_{kl} \quad (4.57)$$

and it is compatible with a deformation parameter λ , which can be regarded as the flow of the renormalisation parameters. This introduces a deviation from general relativity.

The local Hamiltonian constraint, unlike the projectable version of the Hořava’s gravity with only $\int H d^3x = 0$ as non-local Hamiltonian, is quadratic in momentum. Due to the quadratic form $M \equiv \int d^3x (H^2/\sqrt{q}) = 0$ of master constraint which will replace the ordinary Hamiltonian constraint in this framework, this constraint is indeed local because it must vanish at each local point x in order to satisfy the integration. The Hamiltonian constraint takes the form

$$0 = \frac{\sqrt{q}}{2\kappa} H = G_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} + V(q_{ij}) \quad (4.58)$$

$$= -(\beta\pi - \bar{H})(\beta\pi + \bar{H}), \quad (4.59)$$

with V being the potential, where the \bar{H} in the above factorisation is

$$\bar{H}(\tilde{\pi}^{ij}, \bar{q}_{ij}, q) = \sqrt{\bar{G}_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} + V(\bar{q}_{ij}, q)} \quad (4.60)$$

$$= \sqrt{\frac{1}{2}[\bar{q}_{ik}\bar{q}_{jl} + \bar{q}_{il}\bar{q}_{jk}] \tilde{\pi}^{ij} \tilde{\pi}^{kl} + V(q_{ij})}, \quad (4.61)$$

with $\beta^2 = \frac{1}{3(3\lambda-1)}$ and $\kappa = \frac{8\pi G}{c^3}$.

The spatial diffeomorphism constraint is the same as before

$$H_i = -\kappa \nabla_j \tilde{\pi}_i^j = 0. \quad (4.62)$$

The potential is taken to be of the form [1, 10] in this study,

$$V(\bar{q}_{ij}, q) = \frac{-q}{(2\kappa)^2} [R^{(3)} - 2\Lambda_{\text{eff}}], \quad (4.63)$$

where $\Lambda_{\text{eff}} \equiv \frac{3\Lambda}{4a\pi^2}$ with parameter a already introduced in (4.41). This potential corresponds to the low curvature regime of Hořava's gravity. There is a wide class of theories in the high energy regime of Hořava's gravity. We focus on a particular subclass of theories which allow to recover Einstein's operator at weak curvature. This allows us to recover the Wheeler-DeWitt equation as well.

We now proceed to the quantisation of the theory which implies that the constraint becomes an operator and acts on the wave function. In this framework the Hamiltonian constraint is replaced by master constraint. According to [56], it means

$$M \mid \Psi\rangle = 0. \quad (4.64)$$

In the definition of mater constraint $M \equiv \int d^3x (H^2/\sqrt{q}) = 0$ we set $\frac{\sqrt{q}}{2\kappa} H = -(\beta\pi - \bar{H})(\beta\pi + \bar{H})$ using (4.59). Therefore, one obtains $M = \int d^3x \frac{(2\kappa)^2}{q\sqrt{q}} [(\beta\pi + \bar{H})^2(\beta\pi - \bar{H})^2]$. It is sufficient for one of the two terms under this integral to fulfil the constraint (4.64). We can thus choose the form of master constraint to be

$$M \equiv \int d^3x (\beta\pi + \bar{H})^2/\sqrt{q}. \quad (4.65)$$

One could also have picked the other factor, i.e. $(\beta\pi - \bar{H})^2$. One would then obtain the same result but with an opposite sign for the temporal part. This may imply the different choice of time flow. Also if one make such choice, Einstein gravity would be recovered with a sign different; that is the pragmatcal reason why we picked $(\beta\pi + \bar{H})^2$.

Now we have $(\beta\pi + \bar{H})^2 \mid \Psi\rangle = 0$ which implies that $(\beta\pi + \bar{H}) \mid \Psi\rangle = 0$ or $-(\beta\pi + \bar{H}) \mid \Psi\rangle = 0$ is sufficient to fulfil the constraint. We choose the plus sign. The sign choice has also a pragmatic reason: it only affects an overall sign of wave function. The master constraint acting on the wave function thus gives:

$$[\beta\hat{\pi} + \hat{H}(\hat{\pi}^{ij}, \hat{q}_{ij}, \hat{q})] \mid \Psi\rangle = 0 \quad (4.66)$$

and the momentum constraint

$$\hat{H}_i \mid \Psi\rangle = 0. \quad (4.67)$$

In the metric representation, the canonical momenta are realised by $\hat{\pi} = \frac{3\hbar}{i} \frac{\delta}{\delta \ln q}$ and $\hat{\pi}^{ij} = P_{lk}^{ij} \frac{\delta}{\delta \bar{q}_{lk}}$, which operate on $\Psi[\bar{q}_{ij}, q]$ as well. We thus see that the quantum constraint equation has become an equation which resembles a Schrödinger equation (4.68) where the analogue of the time variable is the intrinsic time interval $\delta \ln q^{\frac{1}{3}}$. Thus this model is equipped intrinsic time, when $\lambda > \frac{1}{3}$. Furthermore, unlike the original Wheeler-DeWitt equation being Klein-Gordon form, the first order in time derivative of this new quantum equation guarantees a semi-positive definite probability density $|\Psi[\bar{q}_{ij}, q]|^2$:

$$i\hbar \frac{\delta}{\delta \ln q} \Psi[\bar{q}_{ij}, q] = \frac{\hat{H}(\hat{\pi}^{ij}, \hat{q}_{ij}, \hat{q})}{3\beta} \Psi[\bar{q}_{ij}, q]. \quad (4.68)$$

The momentum constraint

$$\nabla_j \frac{\delta \Psi}{\delta q_{ij}} = 0 \quad (4.69)$$

enforces spatial diffeomorphism symmetry only. We can regard the true local Hamiltonian as $\frac{\bar{H}(x)}{\beta}$, which is not zero, and generates a physical ‘time evolution’ with respect to $\ln q^{\frac{1}{3}}$, which will be shown clearly in 4.3.3. This is deparameterised from the four covariance so that intrinsic time is picked out. Also the time evolution is not frozen within this new framework because the Hamiltonian $\frac{\bar{H}(x)}{\beta}$ is not zero.

The semiclassical Hamilton-Jacobi equation according to [1],

$$\frac{\delta S}{\delta \ln q} = - \frac{\bar{H}(\bar{\pi}^{ij} = P_{kl}^{ij} \frac{\delta S}{\delta \bar{q}_{kl}}; q_{ij})}{3\beta}, \quad (4.70)$$

which is the leading order term of the expansion \hbar . This approximation is similar to the WKB one [200–204] or, in other words, to the optical limit. This should not to be confused with the semiclassical method mentioned in our previous study Chapter 3. Replacing the wave function in (4.68) by the semiclassical ansatz $C \exp \frac{iS}{\hbar}$, with a slowly varying function C , and taking the lowest order of the exponential expansion in \hbar results in the Hamilton-Jacobi equation. All the derivatives acting on C are in higher order. They thus do not appear in the Hamilton-Jacobi equation obtained in the semi-classical limit. The semiclassical Hamilton-Jacobi is of first order in intrinsic time derivative. This implies completeness [205, 206], which means that their integral solutions form a complete set of gauge-invariant, three dimensional diffeomorphism invariant, integration constants of motion.

4.3.3 Heisenberg Picture and Time Evolution

This subsection will discuss some features of physical evolution with respect to global time which is not dependent on the spacetime point x and the Heisenberg picture of this new model is shown.

The local Hamiltonian $\bar{H}(x)/\beta$ replaces the conventional $H(x)$ in the Wheeler-DeWitt equation and produces time translation with respect to $\ln q(x)^{\frac{1}{3}}$, which is dependent on x . To define time rigorously, it would be more adequate to find an x independent degree of freedom in order to avoid a Tomonaga-Schwinger [207, 208] many-fingered time. In addition, one needs to keep in mind that the lapse function N is not arbitrary in this framework. The Hodge decomposition of the 0-form is given by

$$\delta \ln q^{\frac{1}{3}} = \delta h + \nabla_i \delta V^i. \quad (4.71)$$

In this decomposition, δV^i is some vector field and δh is harmonic function and independent of x since any harmonic function on a compact connected Riemannian manifold is a constant as a result from differential geometry (see, for example, [209–211]). It is also gauge invariant under a three dimensional diffeomorphism transformation. Also δV^i being can be gauged away as $\mathcal{L}_{\delta N^i} \ln q^{\frac{1}{3}} = \frac{2}{3} \nabla_i \delta N^i$. Indeed, $\delta \ln q = \frac{\delta q}{q}$ is a scalar and thus suitable to define a *time interval*.

Note that throughout our discussion, $\ln q^{\frac{1}{3}}$ is still the intrinsic time, while h which appears in (4.71) will only be used to show that $\ln q^{\frac{1}{3}}$ contains the information of the x -independent part which can be used to avoid many-fingered time. In general, $\delta \ln q^{\frac{1}{3}}$, i.e. the intrinsic time interval, is always monotonically corresponding to the δh interval since $\delta \ln q^{\frac{1}{3}} = \delta h$ when $\nabla_i \delta V^i$ is gauged away. Thus we can still use $\delta \ln q^{\frac{1}{3}}$ as intrinsic time interval and one can always find its δh to avoid a many-fingered time.

This results in

$$i\hbar \frac{\delta \Psi}{\delta h} = i\hbar \int d^3x \frac{\delta \Psi}{\delta \ln q^{\frac{1}{3}}(x)} \frac{\delta \ln q^{\frac{1}{3}}(x)}{\delta h} \quad (4.72)$$

$$= \left(\int d^3x \frac{\bar{H}(x)}{\beta} \right) \Psi. \quad (4.73)$$

This formula describes the wave function evolving with respect to h which is embedded in $\ln q^{\frac{1}{3}}$. The physical Hamiltonian can be defined as

$$\mathcal{H}_{\text{phys}} \equiv \int d^3x \frac{\bar{H}(x)}{\beta}, \quad (4.74)$$

which is three dimensional diffeomorphism invariant because \bar{H} has a tensor density of weight one. Accordingly, the whole gauge invariant Schrödinger equation is

$$i\hbar \frac{\delta \Psi}{\delta h} = \mathcal{H}_{\text{phys}} \Psi. \quad (4.75)$$

It defines the quantum geometrodynamics in the superspace ${}^{(3)}\mathcal{G}$ with $(\Psi[[q_{ij} \in {}^{(3)}\mathcal{G}], \mathcal{H}_{\text{phys}}, \delta h])$. The time h is x independent and has the same origin as $\ln q^{\frac{1}{3}}$. We will use the latter for most of the discussion.

The time evolution operator can be obtained easily. To do so, one integrates the Schrödinger equation. This is done without any ambiguity since it is x independent rather than many-fingered. The equation (4.75) implies $\delta\Psi = [-\frac{i}{\hbar}\mathcal{H}_{\text{phys}}\Psi]\delta h\Psi$ under an infinitesimal change generated by the gauge invariant time h . Therefore,

$$\Psi[[q_{ij}(h) \in {}^{(3)}\mathcal{G}]] = U(h, h_0)\Psi[[q_{ij}(h_0) \in {}^{(3)}\mathcal{G}]] \quad (4.76)$$

is the *unitary* transformation of the quantum state with the time ordering unitary operator,

$$U(h, h_0) = T \exp[-\frac{i}{\hbar} \int_{h_0}^h \mathcal{H}_{\text{phys}}(h') \delta h']. \quad (4.77)$$

$\mathcal{H}_{\text{phys}}$ is real and gauge invariant and so is δh . Thus, $U(h, h_0)$ is unitary and gauge invariant. This constitutes the Heisenberg picture, which is essential in quantum field theory. Unitary conservation is obvious within this model for quantum gravity.

We shall now study whether this theory of quantum gravity has solutions. We shall look at simplified model with spherical symmetry. First we need to study the spherical symmetric reduction of the full theory.

4.4 Symmetry Reduction

In order to conduct practical calculations, it is necessary to restrict the full symmetry of quantum gravity. We are going to pick a certain subclass of symmetry. It is a standard practice in the canonical quantisation approach to proceed to such a symmetry reduction. We will restrict ourself to the case where the minisuperspace only has spherical symmetry.

The standard method [212] to restrict symmetry is done by using the Lie derivative technique. We consider Lie derivatives acting on the canonical pair $(q_{ij}, \tilde{\pi}^{ij})$,

$$\mathcal{L}_{\xi(a=1,2,3)}q_{ij} = q_{ij,k}\xi^k + \xi^k_{,j}q_{ik} + \xi^k_{,i}q_{kj} = 0 \quad (4.78)$$

$$\mathcal{L}_{\xi(a=1,2,3)}\tilde{\pi}^{ij} = \tilde{\pi}^{ij}_{,k}\xi^k - \xi^j_{,k}\tilde{\pi}^{ik} - \xi^i_{,k}\tilde{\pi}^{kj} + 1 \cdot (\partial_k \xi^k)\tilde{\pi}^{ij} = 0, \quad (4.79)$$

where the 1 in the last term of (4.79) comes from $\tilde{\pi}^{ij}$ having a tensor density weight of one.

The symmetry generator are given by

$$\xi_{(1)} = L_x = i \left[\sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right] \quad (4.80)$$

$$\xi_{(2)} = L_y = i \left[-\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right] \quad (4.81)$$

$$\xi_{(3)} = L_z = i \left[-\frac{\partial}{\partial \phi} \right]. \quad (4.82)$$

These are rotations with respect to the three axes.

We will now work out explicitly the form for $\bar{q}_{ij} \equiv q^{-\frac{1}{3}}q_{ij}$ and $\bar{\pi}^{ij} = q^{\frac{1}{3}}[\tilde{\pi}^{ij} - \frac{q^{ij}}{3}\pi]$ when spherical symmetric variables are used. We first consider \bar{q}_{ij} , its non-zero spherical symmetric components are given by

$$\bar{q}_{\theta\theta} \equiv q^{-\frac{1}{3}}q_a \quad (4.83)$$

$$\bar{q}_{\phi\phi} \equiv q^{-\frac{1}{3}}q_a \sin^2 \theta \quad (4.84)$$

$$\bar{q}_{\alpha\alpha} \equiv q^{-\frac{1}{3}}q_b. \quad (4.85)$$

Thus the reduced form of \bar{q}_{ij} , which is a tensor density of $-\frac{2}{3}$ is given by

$$\bar{q}_{ij} = q^{-\frac{1}{3}} \text{diag}[q_a, q_a \sin^2 \theta, q_b], \quad (4.86)$$

where $q^{-\frac{1}{3}}$ is factored out to make the weight of the fractional tensor density explicit. Note that q_a and q_b are spherical symmetric variables. All variables will in general depend on the radial coordinate α . One sees that q_a is tensor density of weight 0 and q_b is of weight 1.

Its conjugate momentum is $\bar{\pi}^{ij} = q^{\frac{1}{3}}[\tilde{\pi}^{ij} - \frac{q^{ij}}{3}\pi]$, which is a tensor density of $+\frac{5}{3}$. Its spherical symmetric form is given by

$$\bar{\pi}^{ij} = q^{\frac{1}{3}} \text{diag} \frac{1}{2} [P_a \sin \theta, P_a \csc \theta, P_b \sin \theta], \quad (4.87)$$

where $q^{-\frac{1}{3}}$ is factored out explicitly as before.

The other canonical pair is $(\pi, \ln q^{\frac{1}{3}})$. Because $\pi = q_{ij}\tilde{\pi}^{ij} = q_{\theta\theta}\tilde{\pi}^{\theta\theta} + q_{\phi\phi}\tilde{\pi}^{\phi\phi} + q_{\alpha\alpha}\tilde{\pi}^{\alpha\alpha}$ with a scalar density of weight +1, its spherical symmetric form is

$$p \sin \theta \quad (4.88)$$

where p is of weight +1. Its conjugate variable is $\ln q^{\frac{1}{3}}$. When using spherical symmetric variables, q becomes

$$\mathbf{q} \sin^2 \theta \quad (4.89)$$

as it is of weight 2. Notice that the notation \mathbf{q} stands for the variable in its spherical symmetric form. We will often encounter the term $\delta \ln q^{\frac{1}{3}} = \frac{1}{3} \frac{\delta q}{q} = \frac{1}{3} \frac{\sin^2 \theta \delta \mathbf{q}}{\sin^2 \theta \mathbf{q}} = \delta \ln \mathbf{q}^{\frac{1}{3}}$ which is scalar of weight 0; it keeps the same form

$$\delta \ln \mathbf{q}^{\frac{1}{3}} \quad (4.90)$$

in spherical symmetry.

In the spherical symmetry, the symplectic potential (4.48) of the full theory, where every term has a weight of 1, has become

$$\int d^3x \tilde{\pi}^{ij} \delta q_{ij} = \int d^3x (\tilde{\pi}^{ij} \delta \bar{q}_{ij} + \pi \delta \ln q^{\frac{1}{3}}) \quad (4.91)$$

$$= \int d^3x \left\{ \frac{1}{2} \times (2P_a \delta q_a \sin \theta + P_b \delta q_b \sin \theta) + p \sin \theta \delta \ln q^{\frac{1}{3}} \right\}. \quad (4.92)$$

From the symplectic potential, we can determine the correct canonical pairs which are to be used for the canonical quantisation. The conjugate pairs are in Poisson brackets. Note that there is no trace-free projector in the spherical symmetry case as there are no indices to be traced out. The original full theory Poisson brackets (4.49) and (4.50) are

$$\{\bar{q}_{kl}(x), \tilde{\pi}^{ij}(x')\}_{P.B.} = P_{kl}^{ij} \delta(x, x') \quad (4.93)$$

$$\{\ln q^{\frac{1}{3}}(x), \pi(x')\}_{P.B.} = \delta(x, x'), \quad (4.94)$$

and they become⁸

$$\{q_a(\alpha), P_a(\alpha')\}_{P.B.} = \frac{1}{4\pi} \delta(\alpha, \alpha') \quad (4.95)$$

$$\{q_b(\alpha), P_b(\alpha')\}_{P.B.} = \frac{1}{4\pi} 2\delta(\alpha, \alpha') \quad (4.96)$$

$$\{\ln q^{\frac{1}{3}}(\alpha), p(\alpha')\}_{P.B.} = \frac{1}{4\pi} \delta(\alpha, \alpha') \quad (4.97)$$

in spherical symmetric reductions.

Replacing the Poisson brackets with commutator brackets in order to perform canonical quantisation, the commutation relations are

$$[q_a(\alpha), P_a(\alpha')] = i\hbar \frac{1}{4\pi} \delta(\alpha, \alpha') \quad (4.98)$$

$$[q_b(\alpha), P_b(\alpha')] = i\hbar \frac{2}{4\pi} \delta(\alpha, \alpha') \quad (4.99)$$

$$[\ln q^{\frac{1}{3}}(\alpha), p(\alpha')] = \frac{1}{4\pi} i\hbar \delta(\alpha, \alpha'). \quad (4.100)$$

All other combinations of commutator brackets are zero. We find the following operators for the conjugated momentum in the metric representation,

$$\hat{P}_a = \frac{\hbar}{i} \frac{1}{4\pi} \frac{\delta}{\delta q_a} \quad (4.101)$$

$$\hat{P}_b = \frac{\hbar}{i} \frac{2}{4\pi} \frac{\delta}{\delta q_b} \quad (4.102)$$

$$\hat{p} = \frac{\hbar}{i} \frac{3}{4\pi} \frac{\delta}{\delta \ln q}, \quad (4.103)$$

where the 3 on the factor in \hat{p} comes from the exponent of $\ln q^{\frac{1}{3}}$.

⁸where the 4π factor is produced from the integration of $\sin \theta$.

Now we come to the main part of our study, the spherical symmetric Hamiltonian and its wave function. We consider the Wheeler-DeWitt super-Hamiltonian constraint (4.6) valid in the case of Hořava's gravity with $G_{ijkl} = \frac{1}{2}(q_{ik}q_{jl} + q_{il}q_{jk}) - \frac{\lambda}{3\lambda-1}q_{ij}q_{kl}$ where λ is the deformation factor allowing deviations from general relativity. General relativity is recovered in the limit $\lambda = 1$. The constraint takes the form

$$0 = \frac{\sqrt{q}}{2\kappa} H = G_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} + V(q_{ij}) = -(\beta\pi - \bar{H})(\beta\pi + \bar{H}), \quad (4.104)$$

where $\beta = \frac{1}{3(3\lambda-1)}$ and $\kappa = \frac{8\pi G}{c^3}$.

The 'physical' Hamiltonian constraint is

$$\bar{H}(\tilde{\pi}^{ij}, \bar{q}_{ij}, q) = \sqrt{\bar{G}_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} + V(\bar{q}_{ij}, q)} \quad (4.105)$$

$$= \sqrt{\frac{1}{2}(\bar{q}_{ik}\bar{q}_{jl} + \bar{q}_{il}\bar{q}_{jk}) \tilde{\pi}^{ij} \tilde{\pi}^{kl} + V(\bar{q}_{ij}, q)} \quad (4.106)$$

$$= \sqrt{\frac{1}{2}(q^{\frac{1}{3}})(q^{-\frac{1}{3}}) \sin^2 \theta P_a^2 q_a^2 + \frac{1}{4} \sin^2 \theta (q^{\frac{1}{3}})(q^{-\frac{1}{3}}) P_b^2 q_b^2 + V}$$

$$= \sqrt{\frac{1}{2} \sin^2 \theta P_a^2 q_a^2 + \frac{1}{4} \sin^2 \theta P_b^2 q_b^2 + V}, \quad (4.107)$$

where the potential is in terms of Einstein operators because it corresponds to the lower curvature as discussed by [10]. The reason to include only Einstein operators, as explained in 4.3.2, is that in this study our purpose is to understand the quantum solutions to the wave function equation in that regime and its relation to the Wheeler-DeWitt equation. The potential expressed by Einstein operators at lower energies is

$$V = -\frac{q}{(2\kappa)^2} (R^{(3)} - 2\Lambda_{\text{eff}}) \quad (4.108)$$

where $R^{(3)}$ is the three dimensional Ricci scalar and $\Lambda_{\text{eff}} \equiv \frac{3\Lambda}{4a\pi^2}$, as in (4.63).

Let us now consider the case of spherical symmetry where the potential is independent of radial coordiante α ,

$$V = -\frac{q \sin^2 \theta}{(2\kappa)^2} [2\frac{1}{q_a} - 2\Lambda_{\text{eff}}].$$

The $\sin^2 \theta$ comes from the spherical symmetric reduction of q as it is tensor density of weight 2.

The explicit form of the spherical symmetric \bar{H} , with (4.101) and (4.102) in metric representation, is

$$\bar{H}(\tilde{\pi}^{ij}, \bar{q}_{ij}, q) = \sqrt{\frac{1}{2}(\frac{\hbar}{i} \frac{1}{4\pi})^2 \sin^2 \theta q_a^2 \frac{\delta^2}{\delta q_a^2} + \frac{1}{4}(\frac{\hbar}{i} \frac{2}{4\pi})^2 \sin^2 \theta q_b^2 \frac{\delta^2}{\delta q_b^2} + [-\frac{q \sin^2 \theta}{(2\kappa)^2} (2\frac{1}{q_a} - 2\Lambda_{\text{eff}})]}. \quad (4.109)$$

We now use the master constraint (4.66) discussed in Section 4.3. Through quantisation all constraints become operators [186] and act on the wave function. The master and diffeomorphism constraints, (4.67) and (4.66), are respectively

$$[\beta\hat{\pi} + \hat{H}(\hat{\pi}^{ij}, \hat{q}_{ij}, \hat{q})] | \Psi \rangle = 0 \quad (4.110)$$

$$\hat{H}_i | \Psi \rangle = 0. \quad (4.111)$$

Using (4.101), (4.102) and (4.103), we obtain

$$\begin{aligned} & i\hbar \frac{1}{4\pi} \sin \theta \frac{\delta}{\delta \ln \mathbf{q}} \Psi \\ &= \frac{1}{3\beta} \sqrt{\frac{1}{2} \left(\frac{\hbar}{i} \frac{1}{4\pi} \right)^2 q_a^2 \sin^2 \theta \frac{\delta^2}{\delta q_a^2} + \frac{1}{4} \left(\frac{\hbar}{i} \frac{2}{4\pi} \right)^2 q_b^2 \sin^2 \theta \frac{\delta^2}{\delta q_b^2} + \left[-\frac{\mathbf{q} \sin^2 \theta}{(2\kappa)^2} \left(2\frac{1}{q_a} - 2\Lambda_{\text{eff}} \right) \right]} \Psi. \end{aligned}$$

Cancelling $\sin \theta$ on both sides, we obtain a Schrödinger equation-like

$$i\hbar \frac{1}{4\pi} \frac{\delta}{\delta \ln \mathbf{q}} \Psi = \frac{1}{3\beta} \sqrt{\frac{1}{2} \left(\frac{\hbar}{i} \frac{1}{4\pi} \right)^2 q_a^2 \frac{\delta^2}{\delta q_a^2} + \frac{1}{4} \left(\frac{\hbar}{i} \frac{2}{4\pi} \right)^2 q_b^2 \frac{\delta^2}{\delta q_b^2} + V} \Psi. \quad (4.112)$$

This is equivalent of the Wheeler-DeWitt equation in the framework for quantum gravity proposed by Soo and Yu under specifically the spherical symmetry. Note that $\sin \theta$ on both sides of the equation has cancelled out. We will solve the quantum gravitational Schrödinger equation in the next section. Although strictly speaking, the Wheeler-DeWitt equation is a second order differential equation in time, in this framework it becomes a first order differential equation in time. This equation is of the form of a Schrödinger equation. This acquire the benefits that guarantees semi-positivity of probability density because it is of first order of time derivative. In the following, when we use the term Wheeler-DeWitt equation interchangeably with the term quantum gravitational Schrödinger to describe this Schrödinger type equation.

4.5 Wave Function Solution

Finding a solution to the quantum gravitational Schrödinger equation is of great importance as shows that not only the theoretical framework is consistent but it also has solutions which could correspond to the real world as mentioned in the Chapter 1. Investigating solutions of the Wheeler-DeWitt equation for usual General Relativity has a long history see e.g. [191] to find the wave function of universe. The authors of [213], solve the quantum wave function for the standard spherically symmetric Wheeler-DeWitt equation in General Relativity. We shall proceed the same way to find the solution of wave function in Soo and Yu's model. Our quantum wave function solution will have similarities to the

one found in [213]. We will obtain an analytic solution for wave function. We will develop the general formalism for the α -dependent (where α is the radial coordinate introduced above) as well as the α -independent case and then deal with them separately.

We begin by applying the

$$\frac{1}{3\beta} \sqrt{\frac{1}{2} \left(\frac{\hbar}{i} \frac{1}{4\pi} \right)^2 q_a^2 \frac{\delta^2}{\delta q_a^2} + \frac{1}{4} \left(\frac{\hbar}{i} \frac{2}{4\pi} \right)^2 q_b^2 \frac{\delta^2}{\delta q_b^2} + V} \quad (4.113)$$

operator on both sides of the Schrödinger equation

$$\begin{aligned} i\hbar \frac{1}{4\pi} \frac{\delta}{\delta \ln q} \frac{1}{3\beta} \sqrt{\frac{1}{2} \left(\frac{\hbar}{i} \frac{1}{4\pi} \right)^2 q_a^2 \frac{\delta^2}{\delta q_a^2} + \frac{1}{4} \left(\frac{\hbar}{i} \frac{2}{4\pi} \right)^2 q_b^2 \frac{\delta^2}{\delta q_b^2} + V} \Psi = \\ \left\{ \frac{1}{3\beta} \sqrt{\frac{1}{2} \left(\frac{\hbar}{i} \frac{1}{4\pi} \right)^2 q_a^2 \frac{\delta^2}{\delta q_a^2} + \frac{1}{4} \left(\frac{\hbar}{i} \frac{2}{4\pi} \right)^2 q_b^2 \frac{\delta^2}{\delta q_b^2} + V} \right\}^2 \Psi. \end{aligned} \quad (4.114)$$

Clearly when commuting the term (4.113) with $i\hbar \frac{1}{4\pi} \frac{\delta}{\delta \ln q}$, a $\delta(x - x)$ appears. Since the commutator is at the same point, it is effectively a $\delta(0)$. This is a well-known feature of canonical quantisation. DeWitt has explained how to deal with such delta-functions. In [30], he explains that $\delta(x - x)$ should be identified with 0 instead of infinity. To justify this, he models the delta-function by a so-called double peaked delta function.

The detail of DeWitt's method to deal with the $\delta(0)$ term is described in the following. See, for example, eq. (4.19) on page 1121 in [30]. The result is generic and is used systematically in the context of the canonical quantisation method for any two conjugate pair of variables. In our case, these variables are $(\ln q^{\frac{1}{3}}, p)$ where p will be identified with $\frac{\delta}{\delta \ln q}$. We consider their commutator:

$$[\ln q^{\frac{1}{3}}(\alpha), p(\alpha)] = \ln q^{\frac{1}{3}}(\alpha) p(\alpha) - p(\alpha) \ln q^{\frac{1}{3}}(\alpha) = \frac{1}{4\pi} i\hbar \delta(\alpha, \alpha) = \delta(0). \quad (4.115)$$

Since $\delta(0)$ is a c -number, it is thus invariant under the momentum constraint as a diffeomorphism transformation act as

$$\left[\frac{1}{4\pi} i\hbar \delta(0), H_k[N^k] \right] = 0, \quad (4.116)$$

where $H_i[N^i] \equiv \int d^3x N^i(x) H_i(x)$ for an arbitrary function N^i as a standard definition of the momentum constraint. On the other hand, if we apply the commutator of diffeomorphism constraint on left hand side of (4.115), one finds

$$[\ln q^{\frac{1}{3}} p - p \ln q^{\frac{1}{3}}, H_k[N^k]] = -\partial_i ((\ln q^{\frac{1}{3}} p - p \ln q^{\frac{1}{3}}) N^i) = -\frac{1}{4\pi} i\hbar \partial_i (\delta(0) N^i) \quad (4.117)$$

The left hand side of the two equations (4.116) and (4.117) are identical. Therefore, their right hand side are equal and imply $\delta(0) = 0$. As described in [30]: “ *The problem of*

taking commutators of field quantities at the same space-time point therefore never arises with pairs of constraints. ” Furthermore, DeWitt explains that “ This means that the δ -function may, without inconsistency, be thought of as the limit of a sequence of successively narrower twin Peaked functions, all of which are smooth, have unit integral, and vanish at the point $x' = x$ in the valley between the peaks. ” DeWitt suggests a specific construction to fulfil this special shape for the delta function,

$$\delta(x) \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} (f_\epsilon(x - \sqrt{\epsilon}) + f_\epsilon(x + \sqrt{\epsilon}) - 2 \frac{f_\epsilon}{1 + \epsilon}) \quad (4.118)$$

where $f_\epsilon(x) \equiv \epsilon(x^2 + \epsilon^2)^{-1}$. In the limit $\epsilon \rightarrow 0$, the delta function acts as the usual delta function but it takes the value 0 at the point $x = x$.

We thus obtain

$$\begin{aligned} & (i\hbar \frac{1}{4\pi} \frac{\delta}{\delta \ln \mathbf{q}})^2 \Psi = \\ & \frac{1}{9\beta^2} \left\{ \frac{1}{2} \left(\frac{\hbar}{i} \frac{1}{4\pi} \right)^2 q_a^2 \frac{\delta^2}{\delta q_a^2} + \frac{1}{4} \left(\frac{\hbar}{i} \frac{2}{4\pi} \right)^2 q_b^2 \frac{\delta^2}{\delta q_b^2} + V(q_a, \mathbf{q}) \right\} \Psi. \end{aligned} \quad (4.119)$$

To simplify this equation further, we use the following steps. First we observe that the variable is separable with respect to (q_a, \mathbf{q}) and q_b , since the potential contains only (q_a, \mathbf{q}) . Notice that q is treated as independent of q_a and q_b .

In the separation ansatz $\Psi[q_a, q_b, \mathbf{q}] = X[q_a, \mathbf{q}]Y[q_b]$ with C as the universal constant of separation, the wave function is a functional of q_a, q_b and \mathbf{q} whilst it can be separated into functionals of $X[q_a, \mathbf{q}]$ and $Y[q_b]$, respectively. We thus get

$$\left[\frac{1}{9\beta^2} \left\{ \frac{1}{2} \left(\frac{\hbar}{i} \frac{1}{4\pi} \right)^2 q_a^2 \frac{\delta^2}{\delta q_a^2} + V(q_a, \mathbf{q}) \right\} - \left(i\hbar \frac{1}{4\pi} \frac{\delta}{\delta \ln \mathbf{q}} \right)^2 \right] X[q_a, \mathbf{q}] = C X[q_a, \mathbf{q}] \quad (4.120)$$

$$- \frac{1}{4} \left(\frac{\hbar}{i} \frac{2}{4\pi} \right)^2 q_b^2 \frac{\delta^2}{\delta q_b^2} Y[q_b] = C Y[q_b]. \quad (4.121)$$

We shall now solve this functional differential equation. First we shall discuss the case when the variable is α -independent case, i.e., all dynamical variables are independent of the radial coordinate α . In this case, the super-momentum constraint has a trivial contribution and all the functional derivatives become partial derivatives. The (4.120) becomes a second order partial functional differential equation. We make the following definitions: $\ln \mathbf{q} \equiv \tau$, $A \equiv \frac{1}{9\beta^2} \frac{1}{2} \left(\frac{\hbar}{i} \frac{1}{4\pi} \right)^2$, and $B \equiv -\left(i\hbar \frac{1}{4\pi} \right)^2$. The potential term is given by $f[q_a, \tau] \equiv -\frac{1}{9\beta^2} \frac{e^\tau}{(2\kappa)^2} [2\frac{1}{q_a} - 2\Lambda_{\text{eff}}]$. This equation is a hyperbolic equation, and it can be transformed into

$$(-i\sqrt{AB}(\partial_\xi X - \partial_\eta X) + 4B\partial_\xi \partial_\eta X) + f[\xi, \eta] = C X[\xi, \eta], \quad (4.122)$$

after the redefinitions are $\xi \equiv \tau - \frac{Bq_a \ln[q_a]}{\sqrt{-ABq_a^2}}$ and $\eta \equiv \tau + \frac{Bq_a \ln[q_a]}{\sqrt{-ABq_a^2}}$. Then, we assign $\rho \equiv -\frac{2Bq_a\tau}{\sqrt{-ABq_a^2}}$ and $\sigma \equiv 2\tau$, and find

$$-4B\partial_\rho^2 X^2 + 4B\partial_\sigma^2 X^2 - 2i\sqrt{AB}\partial_\rho X + f[\rho, \sigma]X = C X[\rho, \sigma]. \quad (4.123)$$

Making further transformation of X in order to remove the first order derivative terms, we set $u[\rho, \sigma] = X[\rho, \sigma] \exp[-(\frac{i\sqrt{A}}{4\sqrt{B}})\rho + (\frac{i\sqrt{A}}{4\sqrt{B}})\sigma]$, and thus obtain

$$-4B\partial_\rho^2 u^2 + 4B\partial_\sigma^2 u^2 + f[\rho, \sigma]u = C u. \quad (4.124)$$

Then we define $\zeta \equiv \frac{\rho+\sigma}{2}$ and $\chi \equiv \frac{\sigma-\rho}{2}$, and obtain

$$\begin{aligned} C u[\zeta, \chi] = & 4B \frac{\partial}{\partial \zeta \partial \chi} u[\zeta, \chi] - \frac{1}{18\beta^2 \kappa^2} (-\exp[(\frac{1}{4} - i\frac{\sqrt{A}}{4\sqrt{B}})\zeta] \\ & + (\frac{1}{4} + i\frac{\sqrt{A}}{4\sqrt{B}})\chi) + \exp[\frac{\zeta + \chi}{4}\Lambda] u[\zeta, \chi]. \end{aligned} \quad (4.125)$$

We find a solution of the following form, which is similar to that found in [213] and [214],

$$u[\zeta, \chi] = C_3 \exp i(k_\zeta(\zeta)\zeta + k_\chi(\chi)\chi), \quad (4.126)$$

with the respective formulae for k_ζ and k_χ , as dispersion relations for this wave function solution,

$$\begin{aligned} k_\zeta(\zeta) = & \frac{-e^{\frac{\chi}{4}} \Lambda E_{1-\frac{k_\chi}{\chi \partial_\chi k_\chi}}[-\frac{\zeta}{4}] + e^{\frac{1}{4}\chi(1-i\frac{\sqrt{A}}{\sqrt{B}})} E_{1-\frac{k_\chi}{\chi \partial_\chi k_\chi}}[\frac{1}{4}(\frac{i\sqrt{A}}{\sqrt{B}} - 1)\zeta]}{72B\beta^2 \kappa^2 \chi \partial_\chi k_\chi} \\ & + \frac{C}{4Bk_\chi} + c_1 \zeta^{-\frac{k_\chi}{\chi \partial_\chi k_\chi}} \end{aligned} \quad (4.127)$$

$$\begin{aligned} k_\chi(\chi) = & \frac{-e^{\frac{\zeta}{4}} \Lambda E_{1-\frac{k_\zeta}{\zeta \partial_\zeta k_\zeta}}[-\frac{\chi}{4}] + e^{\frac{1}{4}\zeta(1-i\frac{\sqrt{A}}{\sqrt{B}})} E_{1-\frac{k_\zeta}{\zeta \partial_\zeta k_\zeta}}[\frac{1}{4}(\frac{i\sqrt{A}}{\sqrt{B}} - 1)\chi]}{72B\beta^2 \kappa^2 \zeta \partial_\zeta k_\zeta} \\ & + \frac{C}{4Bk_\zeta} + c_2 \chi^{-\frac{k_\zeta}{\zeta \partial_\zeta k_\zeta}}, \end{aligned} \quad (4.128)$$

where c_1 and c_2 are the integration constants and $E_n[x]$ is the exponential integration function. This solution is exact because k_ζ and k_χ are fixed.

The plane wave will occur only when the potential V is zero in (4.112). This corresponds to a Ricci flat space, $R^{(3)} = 0$, with no cosmology constant and meets the physical intuition that the solution is a plane wave when there is no gravitational effect involved. This is different from the usual analysis in full covariance quantum gravity, see for example [214] where the plane wave solutions are also obtained. However, in that case gauge variables are used. The wave function in that situation takes indeed the form of a plane wave in the minisuperspace with k_ζ and k_χ being constant. In our case, only when the

potential is zero, we find that k_ζ as well as k_η are constant. Thus, (4.125) takes a much simpler form

$$4B \frac{\partial}{\partial \zeta \partial \chi} u[\zeta, \chi] = C u[\zeta, \chi], \quad (4.129)$$

which implies that both $k_\zeta = k_\chi = -i \frac{C}{4B}$ are the wave number of the plane wave solution. In both situations, $V \neq 0$ or $V = 0$, we have an exact and analytic solution, which is valuable in many aspects.

The second equation (4.121) can be solved,

$$Y[q_b] = q_b^{\left(\frac{\hbar - i \sqrt{-64\pi^2 C(9\beta^2) - \hbar^2}}{2\hbar}\right)} [C_1 + C_2 q_b^{\frac{i}{\hbar} \sqrt{-64\pi^2 C(9\beta^2) - \hbar^2}}]. \quad (4.130)$$

Secondly we briefly discuss the α -dependent case, where the variables depend on radial coordinate in the spherical symmetric case. The super-momentum constraint is non-trivial and can be obtained as

$$\partial_\alpha \frac{\delta}{\delta q_b} \Psi = 0. \quad (4.131)$$

It means that $\frac{\delta}{\delta q_b} \Psi$ is independent of α and can be set equal to an arbitrary constant k , i.e. $k \equiv \frac{\delta}{\delta q_b} \Psi$. The super-momentum constraint seems to provide only trivial solution. Due to the second functional derivation in (4.121), one obtains zero; thus there is only a trivial solution found in this case. It is still an open question why there is only trivial solution in the α -dependent case in this model. One possible explanation is that in this framework one has already deparametrise the four dimensional diffeomorphism into three dimensional diffeomorphism as described above in Section 4.3. In a sense, this model has a fix wave function such that when the variable has a dependence on α , it must be zero. This might be due to the special character of this new framework. Of course, it is still open question and some particular non-trivial solution might be found. It should be emphasised however that finding an α independent solution is considered none trivial and researchers have published similar results (within different frameworks) in international journals (see, for example, [191, 213, 214]). Also the form of the solution we have found in the α -independence case is similar to their solutions.

4.6 Conclusion

In this chapter we discussed a new approach to a quantum gravity. It is a mixed between Hořava's Gravity and the well established canonical quantisation method. This fusion considered by Soo and Yu is exciting because it could potentially addressed the well known issues of quantum gravity models: the lack of renormalisability, issues with unitarity

and the problem of time. In a spherical symmetric minisuperspace, we calculate the exact analytic solution for the quantum wave function of this newly proposed quantum gravitational theory.

Chapter 5

Conclusion

In this thesis we have considered three different aspects of gravity. We first considered gravitational effects using effective field theories techniques. In particular, we have looked at quantum gravitational threshold effects studied by Ellis and Gaillard within the framework of grand unification. We have demonstrated that the running of the Planck mass can have a non trivial effect and make this effect much more important than naively expected. Indeed, our study of the running of the Planck mass leads to the conclusions that these threshold corrections are much more important than Ellis and Gaillard thought. This result is in line with previous studies of the quantum gravitational threshold corrections to the unification of the coupling constants of the Standard Model.

We have shown that quantum gravity leads a modified fermion mass unification condition in GUTs. The naive unification condition is affected quite significantly by quantum gravitational effects. The running of the Planck mass amplifies the effect of the high order operators. The level of amplification depends on the number of fields in the model considered.

Another possible interpretation of our result is that it is unreliable to predict the unification profile from low energies without further understanding of quantum gravity effect. Without a precise knowledge of the quantum gravitational corrections and full theory of quantum gravity, it is very difficult to draw conclusions about grand unified theories from low energy measurements. Specifically it is hard to check whether fermion masses unify or not. Moreover it is also challenging to predict the gauge coupling constants unification condition.

Another way to look at our results is that quantum gravity offers some leeway to relax the tension between the predictions of grand unified theories and low energy measurements data. An implication of our results is that supersymmetry is not required to unify the gauge

couplings and the Yukawa couplings of the standard model. Quantum gravity can modified the unification condition is such a way that we can unify the standard model in e.g. $SU(5)$ without the need for supersymmetry. This study motivates further understanding for the ultimate quantum theory of gravity.

The next chapter of our thesis deals with a discussion of the frame dependence issues at the quantum level of gravitational theories in the context of a curved spacetime QFT. We found that when mapping from the Jordan frame to the Einstein frame an additional term caused by the change of measure in the partition functional, must be taken into account in order to make the frames equivalent.

We have discuss a systematic method for mapping gravitational theories formulated in the Jordan frame to those in the Einstein frame at the quantum field theory level. We have shown that when going from one frame to the other a boundary terms appears already at the classical level. Furthermore, we have shown that there are some new quantum contributions produced by the Jacobian involved in the field redefinition. This Jacobian originates from the measure in the path integral, hence it is a purely quantum effect. It is properly discussed in the context of quantum fields in curved space-time and this Jacobian can be derived as the expectation value of stress tensor. Its renormalisation is an essential concern in the curved spacetime. The non-trivial Jacobian must be taken into account when we try to make a comparison of physical observables in between frames. Furthermore we also found that this Jacobian term can be derived as a non-conserved current in the quantum level, and alternatively it can also be represented as boundary terms. Therefore the Jacobian related effect may be offset by adding some extra boundary terms, which will not influence the equation of motion in the bulk. We found that this Jacobian may also be interpreted as a non-conserved current. It impacts the original current conservation. This non-conserved current can be compensated by a boundary term. The importance of the Jacobian is shown in both the path integral and canonical quantisation approaches.

The effect of the Jacobian is not only presents in path integral but also importantly in the canonical quantisation approach. The Hamiltonian constraints are the central role of canonical quantisation, within which the variables convert into operators in either the position representation or the momentum representation. It is understood that the constraints is actually originated from the fact of reparameterisation invariance. In addition, they are the generators of certain symmetry transformations (e.g. diffeomorphism invariance). Importantly a Jacobian appears in the Hamiltonian constraint when performing a field reparametrisation of the Hamiltonian in terms of the variables of the other frame.

Again, the Jacobian factor needs to be taken into account when comparing the two frames. In the process of quantisation because all the generalised coordinate and momentum have become operators, the Jacobian factor will be involved in the quantisation as well. Therefore, in both the path integral and canonical quantisation method, the Jacobian effect is important.

From the view point of quantum field theory, it is understood that fields are dummy variables and summed over, in so far as different measures in the respective frames which need to be dealt with. This is much clearer when using the path integral approach to quantise the model. It is noteworthy that field redefinitions cannot affect the calculation of observables since they are physical quantities. The physical equivalence of the frames holds if the field redefinitions are done properly. The result in this study is generic and it may be extended to other types of gravitational theories.

The last facet of quantum gravity consisted in this thesis is a study of a recent proposal for quantum gravity. This new model is a mixed between Hořava's Gravity and the well established canonical quantisation method. This fusion considered by Soo and Yu is exciting because it could potentially resolved the well known issues of quantum gravity models: the lack of renormalisability, issues with unitarity and the problem of time. We have studied this model specifically in a restricted condition. It is the spherical symmetric mini-superspace. We study the analytic solutions for the quantum wave function and found a solution in the case of α independent coordinate.

Within this thesis, we have studied three different aspects of quantum effects in gravity. This study took us from effective field theories, to theories formulated in curved spacetime to finally a fully quantum model of gravity with a quantised spacetime.

Appendix A

Derivation of the Relation Between Trace of the Stress Tensor and the Jacobian

This subsection is to supplement the derivation of (3.92). What follows is true when the current J_ϕ exists but here we show it without J to simplify the expressions. Starting with

$$\langle out, 0 | in, 0 \rangle \equiv Z := \exp \frac{i}{\hbar} W = \int d\mu[\phi] \exp \frac{i}{\hbar} S,$$

then having the Schwinger action principle [73], $\delta \langle out, 0 | in, 0 \rangle = \frac{i}{\hbar} \langle out, 0 | \delta S | in, 0 \rangle$, variation of W is

$$\delta W = \frac{\langle out, 0 | \delta S | in, 0 \rangle}{\langle out, 0 | in, 0 \rangle} = -i\hbar Z^{-1} \delta Z = \frac{\int d\mu[\phi] \delta S \exp \frac{i}{\hbar} S}{\int d\mu[\phi] \exp \frac{i}{\hbar} S} := \langle \delta S \rangle \quad (\text{A.1})$$

where $\delta Z = \frac{i}{\hbar} \int d\mu[\phi] \delta S \exp \frac{i}{\hbar} S$. Also bear in mind the definition of the stress tensor $T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$; then

$$\delta S = \frac{1}{2} \int dV T_{\mu\nu} \delta g^{\mu\nu}$$

and

$$\delta W = \frac{1}{2} \int dV \langle T_{\mu\nu} \rangle \delta g^{\mu\nu} \quad (\text{A.2})$$

where δW is composed by functional integration coming from (A.1). Therefore the difference between the measure $d\mu[\tilde{\phi}_{N'}] = \det C_{N'N} d\mu[\phi_N]$ comes from the Jacobian wherein the boundary terms are abridged for simplicity; the partition functional $Z = \int d\mu[\phi] \exp \frac{i}{\hbar} S$

has become,

$$\begin{aligned} Z' &= \int d\mu[\tilde{\phi}] \exp \frac{i}{\hbar} S \\ &= \int \det C_{N'N} d\mu[\phi_N] \exp \frac{i}{\hbar} S \end{aligned} \quad (\text{A.3})$$

$$\equiv \int (I_{NN'} + c_{NN'}) d\mu[\phi_N] \exp \frac{i}{\hbar} S. \quad (\text{A.4})$$

Also $Z = \exp \frac{i}{\hbar} W$ has become,

$$Z' = \exp \frac{i}{\hbar} W' = \exp \frac{i}{\hbar} (W + \delta W). \quad (\text{A.5})$$

Therefore

$$\delta W = \frac{\hbar}{i} \ln \left(1 + \frac{\int d\mu[\phi] c_{NN'} \exp \frac{i}{\hbar} S}{\int d\mu[\phi] \exp \frac{i}{\hbar} S} \right) \quad (\text{A.6})$$

also because of the previous result (A.2) $\delta W = \frac{1}{2} \int d^4x \langle T_{\mu\nu} \rangle \delta g^{\mu\nu}$; thus equating these two we have

$$\int d\mu[\phi] \exp \frac{i}{\hbar} S \left\{ \left(\exp \frac{i}{\hbar} \frac{1}{2} \int d^4x \langle T_{\mu\nu} \rangle \delta g^{\mu\nu} \right) - 1 \right\} = \int d\mu[\phi] c_{NN'} \exp \frac{i}{\hbar} S \quad (\text{A.7})$$

substituting this into (A.4) we get

$$\begin{aligned} Z' &= \int (1 + \{ \left(\exp \frac{i}{\hbar} \frac{1}{2} \int d^4x \langle T_{\mu\nu} \rangle \delta g^{\mu\nu} \right) - 1 \} d\mu[\phi_N] \exp \frac{i}{\hbar} S \\ &= \int \exp \frac{i}{\hbar} \left(S + \frac{1}{2} \int d^4x \langle T_{\mu\nu} \rangle \delta g^{\mu\nu} \right); \end{aligned}$$

also $Z' = \int \det C_{N'N} d\mu[\phi_N] \exp \frac{i}{\hbar} S$ thus we have

$$i\hbar \ln(\det C_{N'N}) = -\frac{1}{2} \int dV \langle T_{\mu\nu} \rangle (\delta g^{\mu\nu}). \quad (\text{A.8})$$

Now we allow $\delta g^{\mu\nu}$ to become $g^{\mu\nu}$ since it comes from infinitesimal variation of W and if we are only concerned with the integrated value then we can use $g^{\mu\nu}$ instead. After renormalisation of the expectation value then one has proven the relation (3.92)¹,

$$i\hbar \ln(\det C) = -\frac{1}{2} \int dV \langle T_{\mu}^{\mu} \rangle_{\text{ren}}. \quad (\text{A.9})$$

One may notice that our derivation here is more general than in the derivation of the typical conformal anomaly by functional methods (see for example [72, 121–123, 126]). Our derivation does not need to assume (A.3) to be independent of ϕ and factorise this out from the integration. Actually, in our case the field redefinition of ϕ in (3.10) is more complicated than in the conformal case, which chooses $\xi = \frac{1}{6}$ and $\det C_{N'N}$ to not depend on ϕ . Thus the advantage of our formula is that it can accommodate more generic field redefinitions and allow $\det C_{N'N}$ to depend on ϕ .

¹We omit the under-script ‘ren’ hereafter and regard it as understood as the renormalised value.

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